ABELIAN EXTENSIONS OF SEMISIMPLE GRADED CR ALGEBRAS

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ABSTRACT. In this paper we take up the problem of describing the CR vector bundles M over compact standard CR manifolds S, which are themselves standard CR manifolds. They are associated to special graded Abelian extensions of semisimple graded CR algebras.

CONTENTS

1. INTRODUCTION

The Levi–Tanaka algebras were introduced by Noburu Tanaka in [8, 9] to study the differential geometry of CR manifolds. These algebras were further investigated in [3], where also the special class of *standard CR* manifolds associated to them was introduced. In the paper [4] all semisimple Levi–Tanaka algebras were classified, and later it was shown that they correspond to special minimal orbits for actions of

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real Lie groups on flag manifolds (see [5, 6, 10]), which in turn are pre-
cisely the standard CR manifolds that are compact. For nonstandard
homogeneous CR manifolds the problem of classification is still open.
Recently that of classifying the compact homogeneous CR manifolds
of hypersurface type (CR codimension 1) has been solved in [1].

In this paper we take up the problem of describing the CR vector
bundles M over compact standard CR manifolds S, which are them-
selves standard CR manifolds. They are naturally associated to graded
Abelian extensions of semisimple graded CR algebras. These exten-
sions are defined in §2; in §3 we compute the Lie algebra $\mathfrak{g}$ of infini-
tesimal CR automorphisms of M; then, after showing that the partial
complex structure of M is defined by an inner derivation of $\mathfrak{g}$, we ob-
tain necessary and sufficient conditions in order that a graded Abelian
extension of a semisimple graded CR algebra $\mathfrak{s}$ admits a CR structure.
The last section is devoted to the discussion of some examples.

2. Abelian extensions of graded Lie algebras

2.1. **Representations of graded Lie algebras.** Let $\mathfrak{s} = \oplus_{p\in\mathbb{Z}}\mathfrak{s}_p$ be a
finite dimensional $\mathbb{Z}$-graded real Lie algebra. Given a finite dimensional
$\mathfrak{s}$-module $\mathfrak{l}$, the Abelian extension of $\mathfrak{s}$ by $\mathfrak{l}$ is $\mathfrak{s} \oplus \mathfrak{l}$, with Lie product
defined by:

$$[X,Y] = \begin{cases} [X,Y]_\mathfrak{s} & \text{if} \quad X,Y \in \mathfrak{s}\,, \\ X \cdot Y & \text{if} \quad X \in \mathfrak{s}\,,\ Y \in \mathfrak{l} \\ -Y \cdot X & \text{if} \quad Y \in \mathfrak{s}\,,\ X \in \mathfrak{l} \\ 0 & \text{if} \quad X,Y \in \mathfrak{l}\,. \end{cases}$$

We will use interchangeably the words *module* and *representation*.

Given $Y \in \mathfrak{l}$ and $X_1, ..., X_r \in \mathfrak{s}$ we define by recurrence:

$$\begin{cases} [Y] = Y & \text{if} \quad r = 0\,, \\ [X_1, Y] = X_1 \cdot Y & \text{if} \quad r = 1\,, \\ [X_1, X_2, Y] = [X_1, [X_2, Y]] & \text{if} \quad r = 2\,, \\ [X_1, X_2, ..., X_r, Y] = [X_1, [X_2, ..., X_r, Y]] & \text{if} \quad r > 2\,. \end{cases}$$

We call $[X_1, X_2, ..., X_r, Y]$ a *Lie monomial* in Y of length r; we say that
it is *homogeneous* if $X_1, ..., X_r$ are homogeneous elements of $\mathfrak{s}$; homoge-
neous *decreasing* (resp. *increasing*) if moreover $\deg(X_1) \geq \deg(X_2) \geq
\cdots \geq \deg(X_r)$ (resp. $\deg(X_1) \leq \deg(X_2) \leq \cdots \leq \deg(X_r)$); the inte-
ger $\deg(X_1) + \cdots + \deg(X_r)$ is called its *degree*. The monomial $[Y]$ is
homogeneous of degree 0.

Lemma 2.1. *Let $\mathfrak{s}$ be a graded real Lie algebra, $\mathfrak{l}$ an $\mathfrak{s}$-module, and
$Y \in \mathfrak{l}$.*

*The homogeneous decreasing (or increasing) Lie monomials in Y
generate the $\mathfrak{s}$-submodule $\mathfrak{l}'$ of $\mathfrak{l}$ generated by Y.*

Proof. Clearly $\mathfrak{l}'$ is generated by the homogeneous Lie monomials in Y. Thus, to prove our contention, it suffices to show that every Lie monomial in Y, homogeneous of degree d, is a linear combination of homogeneous decreasing (or increasing) Lie monomials in Y of the same degree d. Let $r > 1$, let $[X_1, ..., X_r, Y]$ be a homogeneous Lie monomial and $(i_1, ..., i_r)$ a permutation of the indices $(1, ..., r)$. We claim that $[X_1, ..., X_r, Y] - [X_{i_1}, ..., X_{i_r}, Y]$ is a linear combination of homogeneous Lie monomials in Y of the same degree d and of length $< r$. Since the group of permutations of $\{1, 2, ..., r\}$ is generated by the transpositions exchanging i and $(i+1)$ for $1 \leq i \leq r - 1$, this follows from the formula:

$$[X_1, \ldots, X_i, X_{i+1}, \ldots, X_r, Y] =$$
$$= [X_1, \ldots, X_{i+1}, X_i, \ldots, X_r, Y] + [X_1, \ldots, [X_i, X_{i+1}], \ldots, X_r, Y],$$

where we note that the homogeneous Lie monomials in the right hand side have the same degree of the homogeneous Lie monomial in the left hand side. This proves our claim. The statement of the Lemma follows then by recurrence on the length r of a homogeneous Lie monomial $[X_1, ..., X_r, Y]$.

$\square$

A *gradation* of the $\mathfrak{s}$-module $\mathfrak{l}$ is a decomposition of $\mathfrak{l}$ into a direct sum of finite dimensional vector subspaces: $\mathfrak{l} = \oplus_{p \in \mathbb{Z}} \mathfrak{l}_p$ such that

$$[\mathfrak{s}_p, \mathfrak{l}_q] \subset \mathfrak{l}_{p+q} \qquad \forall p, q \in \mathbb{Z} .$$

If $\mathfrak{a}$ is a graded vector space, we will denote by $\mathfrak{a}_-$ the subspace $\bigoplus_{p<0} \mathfrak{a}_p$ and by $\mathfrak{a}_+$ the subspace $\bigoplus_{p \geq 0} \mathfrak{a}_p$ (note that $\mathfrak{a}_0 \subset \mathfrak{a}_+$ but $\mathfrak{a}_0 \cap \mathfrak{a}_- = \{0\}$). If $\mathfrak{a}$ is a $\mathbb{Z}$-graded Lie algebra, then $\mathfrak{a}_+$ and $\mathfrak{a}_-$ are Lie subalgebras of $\mathfrak{a}$.

A graded representation $\mathfrak{l}$ of the graded Lie algebra $\mathfrak{s}$ is called

- *transitive* if $[\mathfrak{s}_{-1}, Y] \neq \{0\}$ for $Y \in \mathfrak{l}_p \setminus \{0\}$ and $p \geq 0$;
- *nondegenerate* if $[\mathfrak{s}_{-1}, Y] \neq \{0\}$ for $Y \in \mathfrak{l}_{-1} \setminus \{0\}$;
- *fundamental* if $\mathfrak{l}_{p-1} = [\mathfrak{s}_{-1}, \mathfrak{l}_p]$ for $p < 0$, i.e. if $\mathfrak{l}_{-1}$ generates $\mathfrak{l}_-$ as an $\mathfrak{s}_-$-module.

Note that if $\mathfrak{l} \neq \{0\}$ is transitive and fundamental, then $\mathfrak{l}_{-1}$ is different from $\{0\}$.

A graded Lie algebra $\mathfrak{s}$ is called *transitive* (resp. *nondegenerate*, *fundamental*) if it is *transitive* (resp. *nondegenerate*, *fundamental*) when considered as a graded $\mathfrak{s}$-module via the adjoint representation.

If $\mathfrak{s} = \oplus_{p \in \mathbb{Z}} \mathfrak{s}_p$ is a graded Lie algebra, the linear map $D : \mathfrak{s} \to \mathfrak{s}$ defined by

$$D(X) = pX \quad \text{if} \quad p \in \mathbb{Z} \quad \text{and} \quad X \in \mathfrak{s}_p$$

is a derivation of $\mathfrak{s}$. When this derivation D is inner, we say that $\mathfrak{s}$ is *characteristic* and call characteristic any element $E_\mathfrak{s}$ of $\mathfrak{s}$ such that $D = \mathrm{ad}_\mathfrak{s}(E_\mathfrak{s})$.

Lemma 2.2. *Every semisimple graded Lie algebra $\mathfrak{s}$ is characteristic and contains a unique characteristic element $E_\mathfrak{s}$.*

Proof. Indeed, all derivations of a semisimple Lie algebra are inner. Moreover the characteristic element $E_\mathfrak{s}$ is unique because the adjoint representation is faithful.   □

Recall that the *kind* and *co-kind* of a finite dimensional $\mathbb{Z}$-graded Lie algebra $\mathfrak{s} = \oplus_{p \in \mathbb{Z}} \mathfrak{s}_p$ are the integers $\mu = \mu(\mathfrak{s}) = \sup\{p \in \mathbb{Z} \mid \mathfrak{s}_{-p} \neq 0\}$ and $\nu = \nu(\mathfrak{s}) = \sup\{p \in \mathbb{Z} \mid \mathfrak{s}_p \neq 0\}$. If $\mathfrak{s}$ is semisimple, then $\mu(\mathfrak{s}) = \nu(\mathfrak{s})$.

Likewise the kind $\mu(\mathfrak{l})$ and the co-kind $\nu(\mathfrak{l})$ of a finite dimensional graded $\mathfrak{s}$-module $\mathfrak{l} = \oplus_{p \in \mathbb{Z}} \mathfrak{l}_p$ are defined by $\mu(\mathfrak{l}) = \sup\{p \in \mathbb{Z} \mid \mathfrak{l}_{-p} \neq 0\}$ and $\nu(\mathfrak{l}) = \sup\{p \in \mathbb{Z} \mid \mathfrak{l}_p \neq 0\}$.

Lemma 2.3. *Let $\mathfrak{s}$ be a graded Lie algebra, and $\mathfrak{l}$ a graded representation of $\mathfrak{s}$. Then*

(i) $\mathfrak{s}_- \oplus \mathfrak{l}_-$ *is fundamental if and only if $\mathfrak{s}_-$ and $\mathfrak{l}_-$ are both fundamental;*

(ii) *if $\mathfrak{s}$ and $\mathfrak{l}$ are both transitive (resp. nondegenerate) then $\mathfrak{s} \oplus \mathfrak{l}$ is transitive (resp. nondegenerate);*

(iii) *if $\mathfrak{s} \oplus \mathfrak{l}$ is transitive (resp. nondegenerate) then $\mathfrak{l}$ is transitive (resp. nondegenerate);*

Proof. The proof is straightforward.   □

Lemma 2.4. *Assume that $\mathfrak{s}$ is a semisimple graded real Lie algebra. Then every finite dimensional $\mathfrak{s}$-module $\mathfrak{l}$ admits a gradation. If moreover $\mathfrak{l}$ is irreducible, and $\mathfrak{l} = \oplus_{p \in \mathbb{Z}} \mathfrak{l}_p = \oplus_{p \in \mathbb{Z}} \mathfrak{l}'_p$ are two gradations of $\mathfrak{l}$, then there exists an integer k such that $\mathfrak{l}'_p = \mathfrak{l}_{p+k}$.*

Proof. Assume that $\mathfrak{l}$ is irreducible. Let $E_\mathfrak{s}$ be the characteristic element of $\mathfrak{s} = \oplus_{-\mu \leq p \leq \mu} \mathfrak{s}_p$. Then $T : \mathfrak{l} \ni Y \to [E_\mathfrak{s}, Y] \in \mathfrak{l}$ is a semisimple endomorphism with rational eigenvalues. The difference of any two eigenvalues of T is an integer. Hence, if we fix an eigenvalue λ of T and define $\mathfrak{l}_h = \{Y \in \mathfrak{l} \mid [E_\mathfrak{s}, Y] = (\lambda + h)Y\}$ for all $h \in \mathbb{Z}$ we obtain a gradation $\mathfrak{l} = \oplus_{h \in \mathbb{Z}} \mathfrak{l}_h$ of the $\mathfrak{s}$-module $\mathfrak{l}$.

Vice versa, if $\mathfrak{l} = \oplus_{h \in \mathbb{Z}} \mathfrak{l}_h$ is a gradation of the irreducible $\mathfrak{s}$-module $\mathfrak{l}$, the subspaces $\mathfrak{l}_h$ are T-invariant because $E_\mathfrak{s} \in \mathfrak{s}_0$. Hence they are generated by eigenvectors of T. Suppose that $\mathfrak{l}_d$, for some $d \in \mathbb{Z}$, contains a nonzero vector Y. We can assume that Y is an eigenvector of T. If $[E_\mathfrak{s}, Y] = \lambda Y$, we have $\lambda \in \mathbb{Q}$ and, by choosing a basis of the real vector space $\mathfrak{l}$ consisting of homogeneous Lie polynomials in Y, we obtain that $\mathfrak{l}_h = \{Z \in \mathfrak{l} \mid [E_\mathfrak{s}, Z] = (\lambda + h - d)Z\}$.

In general, after decomposing $\mathfrak{l}$ into a direct sum of irreducible $\mathfrak{s}$-submodules, we obtain a gradation of $\mathfrak{l}$ by defining a gradation on each irreducible $\mathfrak{s}$-submodule of the decomposition.

This completes the proof of the lemma. □

Proposition 2.5. *Let $\mathfrak{s}$ be a graded semisimple Lie algebra. Every graded $\mathfrak{s}$-module $\mathfrak{l}$ admits a decomposition into a direct sum of graded irreducible $\mathfrak{s}$-modules. This decomposition is unique up to 0-degree isomorphisms.*

A graded representation of $\mathfrak{s}$ is transitive (resp. nondegenerate, fundamental) if and only if all its irreducible graded components are transitive (resp. nondegenerate, fundamental).

Proof. Let $D : \mathfrak{s} \oplus \mathfrak{l} \to \mathfrak{s} \oplus \mathfrak{l}$ be the derivation of $\mathfrak{s} \oplus \mathfrak{l}$ defined by $D(X) = pX$ if $X \in \mathfrak{s}_p \oplus \mathfrak{l}_p$, for $p \in \mathbb{Z}$. The derivation D commutes with the action of the characteristic element $E_{\mathfrak{s}}$ of $\mathfrak{s}$. We identify $\mathfrak{s}$, via the adjoint representation, to a subalgebra of the Lie algebra $\mathfrak{D}$ of derivations of $\mathfrak{s} \oplus \mathfrak{l}$ and denote by $\mathfrak{s}'$ the Lie subalgebra of $\mathfrak{D}$ generated by $\mathfrak{s}$ and D. Since $D - E_{\mathfrak{s}}$ is 0 on $\mathfrak{s}$, $D - E_{\mathfrak{s}}$ and $\mathfrak{s}$ commute. Thus $\mathfrak{s}'$ is reductive with center $\mathbb{R} \cdot (D - E_{\mathfrak{s}})$.

Next we note that $\mathfrak{l}$ is $\mathfrak{s}'$-invariant and $D - E_{\mathfrak{s}}$ is diagonalizable on $\mathfrak{l}$. Thus the representation of $\mathfrak{s}'$ on $\mathfrak{l}$ is semisimple, and $\mathfrak{l}$ decomposes into a direct sum of D-invariant, i.e. graded, irreducible $\mathfrak{s}$-modules. This decomposition is unique, modulo isomorphisms of $\mathfrak{s}'$-modules. These isomorphisms, commuting with D, are of degree zero.

The last statement is straightforward. □

2.2. **Graded CR algebras and representations.** A *graded CR algebra* is a graded real Lie algebra $\mathfrak{s} = \oplus_{p \in \mathbb{Z}} \mathfrak{s}_p$, with a complex structure $J : \mathfrak{s}_{-1} \to \mathfrak{s}_{-1}$ satisfying:

(i) $J^2 = -\operatorname{Id}_{\mathfrak{s}_{-1}}$;
(ii) $[JX, JY] = [X, Y]$ for every X and Y in $\mathfrak{s}_{-1}$;
(iii) $[A, JX] = J[A, X]$ for every $A \in \mathfrak{s}_0$ and $X \in \mathfrak{s}_{-1}$.

A graded representation $\mathfrak{l}$ of $\mathfrak{s}$ is called a CR *representation* if a complex structure is given on $\mathfrak{l}_{-1}$, that will also be denoted by J, making the Abelian extension $\mathfrak{s} \oplus \mathfrak{l}$ a graded CR algebra.

Lemma 2.6. *Given a graded CR Lie algebra $\mathfrak{s}$ and a nondegenerate graded representation $\mathfrak{l}$, there is at most one CR structure on $\mathfrak{l}$ that makes $\mathfrak{s} \oplus \mathfrak{l}$ into a CR extension of $\mathfrak{s}$.*

Proof. Let $J_{\mathfrak{l}}$ and $J'_{\mathfrak{l}}$ be two complex structure on $\mathfrak{l}_{-1}$ that make $\mathfrak{s} \oplus \mathfrak{l}$ into a CR extension of $\mathfrak{s}$. Then, if $Y \in \mathfrak{l}_{-1}$, we have

$$[J_{\mathfrak{l}} Y, X] = -[Y, J_{\mathfrak{s}} X], \qquad [J'_{\mathfrak{l}} Y, X] = -[Y, J_{\mathfrak{s}} X]$$

for every $X \in \mathfrak{s}_{-1}$, thus

$$[J_{\mathfrak{l}} Y - J'_{\mathfrak{l}} Y, X] = 0$$

for every $X \in \mathfrak{s}_{-1}$. Because of the non degeneracy of $\mathfrak{l}$ we obtain that $J_{\mathfrak{l}}Y = J'_{\mathfrak{l}}Y$ for every $Y \in \mathfrak{l}_{-1}$. $\qquad\square$

Proposition 2.7. *Let $\mathfrak{l}$ be a transitive nondegenerate graded CR representation of a semisimple graded CR algebra $\mathfrak{s}$. Then every graded $\mathfrak{s}$-submodule of $\mathfrak{l}$ is also CR.*

Proof. Let $\mathfrak{l}'$ be a graded $\mathfrak{s}$-submodule of $\mathfrak{l}$, $Y \in \mathfrak{l}'_{-1}$ a nonzero vector and $\mathfrak{l}''$ a graded $\mathfrak{s}$-submodule complementary to $\mathfrak{l}'$. Write $JY = Y' + Y''$ with $Y' \in \mathfrak{l}'$ and $Y'' \in \mathfrak{l}''$. For every $X \in \mathfrak{s}_{-1}$, we have $[X, Y'] + [X, Y''] = [X, JY] = -[JX, Y] \in \mathfrak{l}'$. This implies that $[X, Y''] = 0$ for every $X \in \mathfrak{s}_{-1}$. By the assumption that $\mathfrak{l}$ is non degenerate, we obtain that $Y'' = 0$, and therefore $JY = Y' \in \mathfrak{l}'$. $\qquad\square$

3. Levi–Tanaka extensions

We recall the definition of a Levi–Tanaka algebra. If $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{m}_p$ is a fundamental nondegenerate graded CR algebra then there exists a unique (modulo isomorphisms) transitive graded finite dimensional CR algebra $\mathfrak{g}$, maximal with respect to inclusion, such that $\mathfrak{m} = \mathfrak{g}_-$. This maximal prolongation $\mathfrak{g} = \mathfrak{g}(\mathfrak{m})$ is called a Levi–Tanaka algebra.

In fact we can take $\mathfrak{g} = \oplus_{p \in \mathbb{Z}}\mathfrak{g}_p$ with the subspaces $\mathfrak{g}_p$ recursively defined by:

$$\mathfrak{g}_p = \begin{cases} \mathfrak{m}_p & \text{if} \quad p < 0\,; \\ \{A \in \mathfrak{D}_0(\mathfrak{m}, \mathfrak{m}) \mid A(JX) = JA(X) \; \forall X \in \mathfrak{m}_{-1}\} & \text{if} \quad p = 0\,; \\ \mathfrak{D}_p(\mathfrak{m}, \oplus_{q<p}\mathfrak{g}_q) & \text{if} \quad p > 0\,; \end{cases}$$

where $\mathfrak{D}_p$ denotes degree p homogeneous derivations.

We refer to [8, 9, 3, 5] for a thorough discussion of Levi–Tanaka algebras.

It is worth rehearsing that a transitive semisimple graded CR algebra $\mathfrak{s}$ is a Levi–Tanaka algebra if and only if is nondegenerate. This is equivalent to the fact that every simple ideal of $\mathfrak{s}$ has kind ≥ 2.

Let $\mathfrak{s}$ be a semisimple graded CR algebra and let $\mathfrak{l}$ be an $\mathfrak{s}$-module. Assume that $\mathfrak{l}$ is graded, transitive and CR. The Abelian extension $\mathfrak{s} \oplus \mathfrak{l}$ has a natural structure of graded CR algebra. Under the further assumption that $\mathfrak{s} \oplus \mathfrak{l}$ is fundamental and nondegenerate, we consider the maximal transitive CR prolongation $\mathfrak{g} = \mathfrak{g}(\mathfrak{s} \oplus \mathfrak{l})$ of $\mathfrak{g}_- = \mathfrak{s}_- \oplus \mathfrak{l}_-$. It is a Levi–Tanaka algebra containing $\mathfrak{s} \oplus \mathfrak{l}$.

We will say that $\mathfrak{s} \oplus \mathfrak{l}$ is the *partial Levi–Tanaka extension* of $\mathfrak{s}$ by $\mathfrak{l}$ and $\mathfrak{g}$ is the *Levi–Tanaka extension* of $\mathfrak{s}$ by $\mathfrak{l}$.

3.1. **The structure of the Levi–Tanaka prolongation.** Let $\mathfrak{s}$ be a semisimple graded CR algebra, and $\mathfrak{l}$ a graded, transitive, CR, fundamental and nondegenerate $\mathfrak{s}$-module. Let $\mathfrak{g} = \oplus_{p \in \mathbb{Z}}\mathfrak{g}_p$ be the Levi–Tanaka extension of $\mathfrak{s}$ by $\mathfrak{l}$. Denote by

$$\pi : \mathfrak{s} \oplus \mathfrak{l} \to \mathfrak{l}$$

the projection onto $\mathfrak{l}$ along $\mathfrak{s}$.

Lemma 3.1. π *is a* 0*-degree derivation of* $\mathfrak{s} \oplus \mathfrak{l}$*, commuting with* $\mathfrak{s}_0$*.*

Since $\pi(\mathfrak{g}_p) = \mathfrak{l}_p \subset \mathfrak{g}_p$ for all $p < 0$ and π commutes with the partial complex structure J on $\mathfrak{g}_{-1} = \mathfrak{s}_{-1} \oplus \mathfrak{l}_{-1}$, there is a unique element in $\mathfrak{g}_0$, that we still denote by π, such that $[\pi, X] = \pi(X)$ for all $X \in \mathfrak{s} \oplus \mathfrak{l}$.

Lemma 3.2. $\mathrm{ad}_{\mathfrak{g}}(\pi) : \mathfrak{g} \to \mathfrak{g}$ *is semisimple.*

Proof. Indeed $\mathrm{ad}_{\mathfrak{g}}(\pi)|_{\mathfrak{g}_-}$ is semisimple and hence the statement follows from Lemma 3.8 in [3]. $\square$

We say that a partial Levi–Tanaka extension $\mathfrak{s} \oplus \mathfrak{l}$ is *semisimple* if the corresponding Levi–Tanaka extension $\mathfrak{g}(\mathfrak{s} \oplus \mathfrak{l})$ is semisimple, and that $\mathfrak{s} \oplus \mathfrak{l}$ and $\mathfrak{g}(\mathfrak{s} \oplus \mathfrak{l})$ are *proper* if $\mathfrak{g}(\mathfrak{s} \oplus \mathfrak{l})$ does not contain any semisimple ideal.

Proposition 3.3. *Let* $\mathfrak{s}$ *be a semisimple fundamental graded CR algebra and* $\mathfrak{l}$ *a transitive fundamental graded CR $\mathfrak{s}$-module, such that* $\mathfrak{s} \oplus \mathfrak{l}$ *is nondegenerate. Let* $\mathfrak{g}$ *be the Levi–Tanaka extension of* $\mathfrak{s}$ *by* $\mathfrak{l}$*.*
Then there exist two graded CR ideals $\mathfrak{s}'$ *and* $\mathfrak{s}''$ *of* $\mathfrak{s}$ *and two graded CR $\mathfrak{s}$-submodules* $\mathfrak{l}'$ *and* $\mathfrak{l}''$ *of* $\mathfrak{l}$ *such that:*

  (i) $\mathfrak{s} = \mathfrak{s}' \oplus \mathfrak{s}''$, $\mathfrak{l} = \mathfrak{l}' \oplus \mathfrak{l}''$*;*
  (ii) $\mathfrak{s}' \oplus \mathfrak{l}'$ *and* $\mathfrak{s}'' \oplus \mathfrak{l}''$ *are ideals in* $\mathfrak{s} \oplus \mathfrak{l}$*;*
  (iii) $\mathfrak{g} = \mathfrak{g}(\mathfrak{s}' \oplus \mathfrak{l}') \oplus \mathfrak{g}(\mathfrak{s}'' \oplus \mathfrak{l}'')$ *as a sum of ideals;*
  (iv) $\mathfrak{g}(\mathfrak{s}' \oplus \mathfrak{l}')$ *is a semisimple Levi–Tanaka extension of* $\mathfrak{s}'$*;*
  (v) $\mathfrak{g}(\mathfrak{s}'' \oplus \mathfrak{l}'')$ *is a proper Levi–Tanaka extension of* $\mathfrak{s}''$*.*

Proof. Let $\mathfrak{r}$ be the radical of $\mathfrak{g}$. Since $\mathfrak{r} \cap \mathfrak{s} = 0$, the map $\mathrm{ad}(\pi)|_{\mathfrak{r}_-} : \mathfrak{r}_- \to \mathfrak{l}_-$ is injective. This implies that $\mathfrak{r}_-$ is contained in $\mathfrak{l}_-$. Let $\mathfrak{l}'' = \mathfrak{l} \cap \mathfrak{r}$, so that $\mathfrak{l}''_- = \mathfrak{r}_-$.

Let $\mathfrak{L}$ be a graded CR Levi subalgebra of $\mathfrak{g}$, containing $\mathfrak{s}$ and $\mathrm{ad}_{\mathfrak{g}}(\pi)$-invariant (its existence is granted by [6]). Let $\mathfrak{l}' = \mathfrak{L} \cap \mathfrak{l}$. Then $\mathfrak{l} = \mathfrak{l}' \oplus \mathfrak{l}''$, because $\mathfrak{l}_- = \mathfrak{l}'_- \oplus \mathfrak{l}''_-$ and every irreducible component of $\mathfrak{l}$ is generated by its degree -1 homogeneous elements. Every simple ideal $\mathfrak{a}$ of $\mathfrak{s}$ is contained in a simple ideal $\mathfrak{L}_{\mathfrak{a}}$ of $\mathfrak{L}$. Define

$$\mathfrak{s}' = \bigoplus \{\mathfrak{a} \mid \mathfrak{a} \text{ is a simple ideal of } \mathfrak{s} \text{ and } \mathfrak{L}_{\mathfrak{a}} \cap \mathfrak{l} \neq 0\} \oplus \ker_{\mathfrak{s}} \mathfrak{l}$$

and

$$\mathfrak{s}'' = \bigoplus \{\mathfrak{a} \mid \mathfrak{a} \text{ is a simple ideal of } \mathfrak{s}, \ \mathfrak{L}_{\mathfrak{a}} \cap (\mathfrak{l} \oplus \ker_{\mathfrak{s}} \mathfrak{l}) = 0\}.$$

Then condition (i) is fulfilled and condition (ii) follows because $[\mathfrak{s}', \mathfrak{l}''] = 0$ and $[\mathfrak{s}'', \mathfrak{l}'] = 0$; condition (iii) is a consequence of (ii) and Proposition 3.3 in [3]; condition (iv) holds because $\mathfrak{g}(\mathfrak{s}' \oplus \mathfrak{l}') = \bigoplus_{\mathfrak{a} \subset \mathfrak{s}'} \mathfrak{L}_{\mathfrak{a}}$ is a direct sum of simple ideals.

Finally, assume by contradiction that $\mathfrak{g}(\mathfrak{s}'' \oplus \mathfrak{l}'')$ contains a a simple ideal $\mathfrak{a}$. Then $[\mathfrak{a}, \mathfrak{l}''] = 0$ because $\mathfrak{l}'' \subset \mathfrak{r}$, and $[\mathfrak{a}, \mathfrak{l}'] = 0$ because $\mathfrak{a}$ is not

contained in $\mathfrak{s}'$. Hence $\mathfrak{a} \subset \ker_\mathfrak{s} \mathfrak{l} \subset \mathfrak{s}'$ and we get a contradiction. The proof is complete. □

Example 3.4. Let

$$\mathfrak{s} = \mathfrak{sl}(2, \mathbb{C}) = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| a_{ij} \in \mathbb{C}, \ a_{11} + a_{22} = 0 \right\}.$$

We consider the gradation $\mathfrak{s} = \oplus_{-1 \le p \le 1} \mathfrak{s}_p$ corresponding to the characteristic element

$$E_\mathfrak{s} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

and the CR structure defined by the matrix

$$J_\mathfrak{s} = \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}.$$

We consider the space $\mathfrak{l}$ of anti-Hermitian 2×2 matrices and we let $\mathfrak{s}$ act on $\mathfrak{l}$ by:

$$[X, A] = X \cdot A = AX + X^*A \qquad \text{for } X \in \mathfrak{s}, \ A \in \mathfrak{l}.$$

We define on $\mathfrak{l}$ the gradation corresponding to the characteristic element

$$[E, A] = [E_\mathfrak{s}, A] - A \quad \text{for } A \in \mathfrak{l}$$

and the CR structure obtained by restriction to $\mathfrak{l}_{-1}$ of the action of $J_\mathfrak{s}$ on $\mathfrak{l}$.

Define $\mathfrak{su}(2, 2)$ as the simple Lie algebra of 4×4 complex matrices Y satisfying

$$Y I_{2,2} + I_{2,2} Y^* = 0$$

for

$$I_{2,2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then $\mathfrak{s} \oplus \mathfrak{l}$ is isomorphic to the Lie subalgebra of $\mathfrak{su}(2, 2)$ consisting of matrices of the form

$$\begin{pmatrix} X & A \\ 0 & -X^* \end{pmatrix} \qquad \text{with } X \in \mathfrak{s} \text{ and } A \in \mathfrak{l}.$$

and clearly $\mathfrak{g}(\mathfrak{s} \oplus \mathfrak{l}) = \mathfrak{su}(2, 2)$ is semisimple.

Example 3.5. Let $\mathfrak{s} = \mathfrak{sl}(2, \mathbb{C})$ with the same gradation and CR structure as in the previous example and let $\mathfrak{l} = \mathfrak{l}' \oplus \mathfrak{l}''$ where both $\mathfrak{l}'$ and $\mathfrak{l}''$ are isomorphic to the representation $\mathfrak{l}$ of the previous example, with the same gradation and the same CR structure.

In this case $\mathfrak{g}(\mathfrak{s} \oplus \mathfrak{l}) = \mathfrak{s} \oplus \mathfrak{l} \oplus \mathbb{R}\pi' \oplus \mathbb{R}\pi''$, where π' and π'' are projections onto $\mathfrak{l}'$ and $\mathfrak{l}''$ respectively. In this case $\mathfrak{g}$ is proper. It is the Levi–Tanaka algebra of kind two and CR codimension two, associated to a singular pencil of 3×3 Hermitian matrices (see [7]).

Next we give a sufficient condition for a Levi–Tanaka extension to be proper.

Lemma 3.6. *Let $\mathfrak{g}$ be a finite dimensional Levi–Tanaka algebra and let $\mathfrak{s}$ be a semisimple Levi–Tanaka subalgebra of $\mathfrak{g}$. Let $\varphi_- : \mathfrak{g}_- \to \mathfrak{s}_-$ be a homomorphism of graded CR algebras, whose restriction to $\mathfrak{s}_-$ is the identity. Then φ_- extends to a split surjective homomorphism of graded CR algebras*

$$\tilde{\varphi} : \mathfrak{g} \to \mathfrak{s} \,.$$

Proof. For $A \in \mathfrak{g}_0$, the map $\varphi_0(A) : \mathfrak{s}_- \to \mathfrak{s}_-$ defined by:

$$\varphi_0(A)(X) = \varphi_-([A, X]) \qquad \forall X \in \mathfrak{s}_-$$

is a degree zero derivation of $\mathfrak{s}_-$, that commutes with the complex structure $J_\mathfrak{s}$ on $\mathfrak{s}_{-1}$, and hence defines an element of $\mathfrak{s}_0 \subset \mathfrak{D}_0(\mathfrak{s}_-, \mathfrak{s}_-)$. This gives a homomorphism $\varphi_0 : \mathfrak{g}_0 \to \mathfrak{s}_0$.

Next we define recursively $\varphi_p : \mathfrak{g}_p \to \mathfrak{s}_p$, for $p > 0$, by

$$\varphi_p(A)(X) = \varphi_{p-h}([A, X]) \qquad \forall A \in \mathfrak{g}_p, \quad \forall X \in \mathfrak{s}_{-h}, \quad h > 0 \,.$$

One verifies that $\varphi(A)$ is an element of $\mathfrak{s}_p = \mathfrak{D}_p(\mathfrak{s}_-, \oplus_{q<p}\mathfrak{s}_q)$.

Define $\tilde{\varphi} : \mathfrak{g} \to \mathfrak{s}$ to be equal to φ_- on $\mathfrak{g}_-$ and to φ_p on $\mathfrak{g}_p$ for $p \geq 0$. Since $\tilde{\varphi}$ is the identity on $\mathfrak{s} \subset \mathfrak{g}$, the map $\tilde{\varphi} : \mathfrak{g} \to \mathfrak{s}$ is onto and splits.

It is clear that $\tilde{\varphi}$ is a surjective homomorphism of graded CR algebras. □

Proposition 3.7. *If $\mathfrak{s}$ is a semisimple Levi–Tanaka algebra and $\mathfrak{l}$ a nondegenerate faithful graded CR $\mathfrak{s}$-module, then the Levi–Tanaka extension $\mathfrak{g}$ of $\mathfrak{s}$ by $\mathfrak{l}$ is proper.*

Proof. From the previous lemma we know that there is a graded CR ideal $\mathfrak{b}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{b}$. Since $[\pi, \mathfrak{b}] \subset \mathfrak{b} \cap \mathfrak{l}$, we obtain that $\mathfrak{l}_-$, and hence $\mathfrak{l}$, are contained in $\mathfrak{b}$. Assume by contradiction that $\mathfrak{g}$ contains a simple ideal $\mathfrak{a}$.

If $[\pi, \mathfrak{a}] \neq \{0\}$ or $\mathfrak{a} \cap \mathfrak{b} \neq \{0\}$ then $\mathfrak{a} \subset \mathfrak{b}$. Therefore $\mathfrak{a}_{-1} \subset \mathfrak{l}_{-1}$ and $[\mathfrak{a}_{-1}, \mathfrak{s}_{-1} \oplus \mathfrak{l}_{-1}] \subset [\mathfrak{a}, \mathfrak{s}] + [\mathfrak{l}, \mathfrak{l}] = \{0\}$. By the condition that $\mathfrak{s}_- \oplus \mathfrak{l}_-$ is nondegenerate, we obtain that $\mathfrak{a}_{-1}$, and hence $\mathfrak{a}_-$, are $\{0\}$. This contradicts the transitivity of $\mathfrak{g}$.

If $\mathfrak{a} \cap \mathfrak{b} = \{0\}$ and $[\pi, \mathfrak{a}] = 0$ then $\mathfrak{a}_- \subset \mathfrak{s}_-$, but this contradicts the faithfulness of $\mathfrak{l}$, because $[\mathfrak{a}, \mathfrak{b}] = \{0\}$. □

If $\mathfrak{g}$ is a proper Levi–Tanaka extension of $\mathfrak{s}$ by a transitive fundamental nondegenerate graded CR $\mathfrak{s}$-module $\mathfrak{l}$, Proposition 3.3 and Lemma 3.6 yield a decomposition:

$$\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{b} = \mathfrak{s} \oplus \mathfrak{a} \oplus \mathfrak{r}$$

where $\mathfrak{b}$ is an ideal containing $\mathfrak{l}$, which is the direct sum of the radical $\mathfrak{r}$ of $\mathfrak{g}$ and of a semisimple $\mathfrak{a} \subset \mathfrak{g}_0$ that is an ideal in the Levi subalgebra $\mathfrak{L} = \mathfrak{s} \oplus \mathfrak{a}$ of $\mathfrak{g}$.

Denote by $T : \mathfrak{b} \to \mathfrak{b}$ the restriction of $\mathrm{ad}_\mathfrak{g}(\pi)$ to $\mathfrak{b}$. Since T is semisimple, its minimal polynomial $m_T(\lambda)$ is a product $m_T(\lambda) = p_1(\lambda) \cdots p_t(\lambda)$ of irreducible distinct polynomials in $\mathbb{R}[\lambda]$. Set:

$$\mathfrak{b}^{p_i} = \{X \in \mathfrak{b} \mid p_i(T)(X) = 0\} \qquad \text{for} \quad i = 1, ..., t\,.$$

Being $\mathfrak{l} \subset \mathfrak{b}$, we can assume that $\mathfrak{l} \subset \mathfrak{b}^{p_1}$, with $p_1(\lambda) = \lambda - 1$. Moreover, all $\mathfrak{b}^{p_i}$ are graded, with kind $\mu(\mathfrak{b}^{p_i}) \leq 0$ if $i \geq 2$, because $\mathfrak{b}_- = \mathfrak{l}_- \subset \mathfrak{b}^{p_1}$. The $\mathfrak{b}^{p_i}$'s are $\mathfrak{s}$-modules, as T commutes with the restriction of $\mathrm{ad}_\mathfrak{g}(\mathfrak{s})$ to $\mathfrak{b}$. For every $2 \leq i \leq t$, the $\mathfrak{s}$-module $\mathfrak{b}^{p_i}$ contains an element U homogeneous of minimal degree $d \geq 0$. Hence $[U, \mathfrak{s}_{-1}] = \{0\}$ and there exists $Y \in \mathfrak{l}_{-1}$ such that $[U, Y] \neq 0$. This implies that for each i with $2 \leq i \leq t$, also the polynomial $p_i(\lambda - 1)$ is a factor of the minimal polynomial $m_T(\lambda)$ of T, because $p_i(T - I)([U, Y]) = 0$. From this observation we deduce that all irreducible factors of $m_T(\lambda)$ are of the first order and T can be diagonalized: its eigenvalues are integers h with $2 - t \leq h \leq 1$. Thus we rewrite:

$$\mathfrak{b} = \bigoplus_{2-t \leq h \leq 1} \mathfrak{b}^h$$

where

$$\mathfrak{b}^h = \{X \in \mathfrak{b} \mid [\pi, X] = hX\}, \qquad h \in \mathbb{Z},\ 2 - t \leq h \leq 1$$

are the eigenspaces of T.

Moreover, $\mathfrak{b}^1 = \mathfrak{l}$. Indeed, if this were not the case, there would be a nonzero graded $\mathfrak{s}$-submodule $\mathfrak{w}$ of $\mathfrak{b}^1$ which is a complement of $\mathfrak{l}$ in $\mathfrak{b}^1$. Take a nonzero element W homogeneous of minimal degree in $\mathfrak{w}$. Then, because $\mathfrak{g}$ is transitive, there is $X \in \mathfrak{l}_{-1}$ with $[W, X] \neq 0$. But this gives a contradiction, because $T([W, X]) = 2[W, X]$, and all eigenvalues of T are ≤ 1.

The ideal $\mathfrak{b}$ of $\mathfrak{g}$ contains the radical $\mathfrak{r}$ of $\mathfrak{g}$. In particular the radical $\mathfrak{r}(\mathfrak{b})$ of $\mathfrak{b}$ coincides with the radical $\mathfrak{r}$ of $\mathfrak{g}$. Thus the decomposition

$$\mathfrak{b} = \mathfrak{a} \oplus \mathfrak{r}$$

is a Levi–Malcev decomposition of $\mathfrak{b}$. Since π is a semisimple element of $\mathfrak{b}_0$, we can choose $\mathfrak{a} \subset \mathfrak{b}_0$ to be π-invariant. Thus we decompose $\mathfrak{a}$ into a direct sum of eigenspaces of T: $\mathfrak{a} = \oplus_{h \leq 0} \mathfrak{a}^h$. Since $\oplus_{h < 0} \mathfrak{a}^h$ is a nilpotent ideal of $\mathfrak{a}$, it is $\{0\}$ and we obtain $\mathfrak{a} = \mathfrak{a}^0$, i.e. $[\pi, \mathfrak{a}] = 0$.

We have obtained that

$$[\mathfrak{a}, \mathfrak{s}] = \{0\} \quad \text{and} \quad [\mathfrak{a}, \mathfrak{l}] \subset \mathfrak{l}\,.$$

Therefore $\mathfrak{a}$ is a semisimple subalgebra of the algebra of zero degree $\mathfrak{s}$-endomorphisms of $\mathfrak{l}$. The discussion beolow will show that actually $\mathfrak{a}$ is a Levi subalgebra of the algebra of zero degree $\mathfrak{s}$-endomorphisms of $\mathfrak{l}$.

Now observe that $\mathfrak{r}$ (considered as a subalgebra of $\mathfrak{gl}_\mathbb{R}(\mathfrak{g})$ via the adjoint representation) is splittable and therefore decomposes into the

direct sum of the ideal $\mathfrak{n}(\mathfrak{g})$ consisting of its nilpotent elements and of a maximal Abelian subalgebra $\mathfrak{t}$ of $\mathfrak{r}(\mathfrak{g})$ consisting of semisimple elements. Since $[\pi, \mathfrak{L}] = \{0\}$, the semisimple element π belongs to $\mathfrak{t}$. Thus we can take $\pi \in \mathfrak{t}$ and $\mathfrak{t} \subset \mathfrak{g}_0$.

The nilpotent ideal $\mathfrak{n}(\mathfrak{g})$ is $\mathrm{ad}_{\mathfrak{g}}(\pi)$-invariant and therefore decomposes into eigenspaces of T:

$$\mathfrak{n}(\mathfrak{g}) = \bigoplus_{\substack{h \in \mathbb{Z} \\ h \leq 0}} \mathfrak{n}^h(\mathfrak{g})$$

where $\mathfrak{n}^h(\mathfrak{g}) = \{X \in \mathfrak{n}(\mathfrak{g}) \,|\, [\pi, X] = hX\}$. Moreover $\mathfrak{n}^1(\mathfrak{g}) = \mathfrak{l}$ and $\mathfrak{n}^0(\mathfrak{g})$ is a subalgebra and an $\mathfrak{s}$-submodule, with $[\mathfrak{n}^0(\mathfrak{g}), \mathfrak{l}] \subset \mathfrak{l}$.

The irreducible case. Now we suppose that $\mathfrak{l}$ is irreducible. In this case we know from Ch. 9 of [2] that for $\mathfrak{a}$ we have the cases:

(i) the representation $\mathfrak{l}$ is of the real or of the complex type: then $\mathfrak{a} = \{0\}$;

(ii) the representation $\mathfrak{l}$ is of the quaternionic type: then $\mathfrak{a} \simeq \mathfrak{su}(2)$.

By Engel's theorem the set of $Y \in \mathfrak{l}$ such that $[\mathfrak{n}^0(\mathfrak{g}), Y] = 0$ is different from $\{0\}$. Since it is an $\mathfrak{s}$-submodule of $\mathfrak{l}$, and we are supposing that $\mathfrak{l}$ is irreducible, it must coincide with $\mathfrak{l}$. From this we derive that $[\mathfrak{n}^0(\mathfrak{g}), \mathfrak{l}] = \{0\}$. But $\mathfrak{n}^0(\mathfrak{g})$ is graded and $\mathfrak{n}^0(\mathfrak{g}) = \bigoplus_{p \geq 0} \mathfrak{n}_p^0(\mathfrak{g})$. For a nonzero homogeneous term U of minimal degree in $\mathfrak{n}^0(\mathfrak{g})$ we would have $[U, \mathfrak{s}_{-1}] = \{0\}$ and $[U, \mathfrak{l}_{-1}] = 0$, contradicting the transitivity of $\mathfrak{g}$. Thus $\mathfrak{n}^0(\mathfrak{g}) = \{0\}$ and this implies that $\mathfrak{n}^h(\mathfrak{g}) = \{0\}$ for all $h \leq 0$.

Hence we obtain $\mathfrak{n}(\mathfrak{g}) = \mathfrak{l}$ and $\mathfrak{t} = \mathfrak{r}^0(\mathfrak{g})$ consists of semisimple elements. We have $\mathfrak{t} = \mathbb{R}\pi$ if $\mathfrak{l}$ is of the real or of the quaternionic type and $\mathfrak{t} = \mathbb{C}\pi$ if $\mathfrak{l}$ is of the complex type.

We have proved the following:

Theorem 3.8. *Let $\mathfrak{s}$ be a semisimple graded CR algebra. Let $\mathfrak{l}$ be an irreducible nondegenerate CR graded $\mathfrak{s}$-module. Assume that the Levi–Tanaka extension $\mathfrak{g}$ of $\mathfrak{s}$ by $\mathfrak{l}$ is proper. Then $\mathfrak{g}$ is a finite dimensional real Lie algebra and we have:*

(i) if $\mathfrak{l}$ is of the real type, then $\mathfrak{s}$ is a Levi subalgebra of $\mathfrak{g}$, and $\mathfrak{g}$ admits the CR Levi-Malcev decomposition:

$$\mathfrak{g} = \mathfrak{s} \oplus (\mathfrak{l} \oplus \mathbb{R}\pi),$$

where $\mathfrak{l}$ is the maximal nilpotent ideal of the radical $\mathfrak{r}$ of $\mathfrak{g}$ and $\mathbb{R}\pi$ a maximal Abelian subalgebra of semisimple elements of $\mathfrak{r}$;

(ii) if $\mathfrak{l}$ is of the complex type, then $\mathfrak{s}$ is a Levi subalgebra of $\mathfrak{g}$, and $\mathfrak{g}$ admits the CR Levi-Malcev decomposition:

$$\mathfrak{g} = \mathfrak{s} \oplus (\mathfrak{l} \oplus \mathbb{C}\pi),$$

where $\mathfrak{l}$ is the maximal nilpotent ideal of the radical $\mathfrak{r}$ of $\mathfrak{g}$ and $\mathbb{C}\pi$ a maximal Abelian subalgebra of semisimple elements of $\mathfrak{r}$;

(*iii*) *If $\mathfrak{l}$ is of the quaternionic type, $\mathfrak{g}$ has a CR Levi-Malcev decomposition:*

$$\mathfrak{g} = (\mathfrak{s} \oplus \mathfrak{a}) \oplus (\mathfrak{l} \oplus \mathbb{R}\pi),$$

with a Levi subalgebra $\mathfrak{L} = \mathfrak{s} \oplus \mathfrak{a}$ consisting of the direct sum of $\mathfrak{s}$ and of a simple ideal $\mathfrak{a} \subset \mathfrak{g}_0$, isomorphic to $\mathfrak{su}(2)$; $\mathfrak{l}$ is the maximal nilpotent ideal of the radical $\mathfrak{r}$ of $\mathfrak{g}$, and $\mathbb{R}\pi$ is a maximal Abelian subalgebra of semisimple elements of $\mathfrak{r}$.

The general case. If the representation $\mathfrak{l}$ is not irreducible, the structure of $\mathfrak{g}$ may be more complicated (see the example below).

Decompose $\mathfrak{l}$ into a direct sum of irreducible CR graded representations:

$$\mathfrak{l} = \bigoplus_i m_i \cdot \mathfrak{l}^i$$

where the m_i's are positive integers and the $\mathfrak{l}^i$'s irreducible graded CR representations of $\mathfrak{s}$, pairwise not isomorphic as graded representations (i.e. distinct $\mathfrak{l}^i$'s can be isomorphic, but endowed with non isomorphic gradations).

We keep the notation used throughout this section. In particular, $\mathfrak{a} \subset \mathfrak{g}_0$ is the Levi part of the ideal $\mathfrak{b}$ complementary to $\mathfrak{s}$ in $\mathfrak{g}$ and $\mathfrak{t} \subset \mathfrak{g}_0$ is a maximal Abelian subalgebra of semisimple elements of the radical $\mathfrak{r}$ of $\mathfrak{g}$.

Then

$$\mathfrak{a} \oplus \mathfrak{t} = \bigoplus_i (\mathfrak{a}_i \oplus \mathfrak{t}_i)$$

where $(\mathfrak{a}_i \oplus \mathfrak{t}_i)$ is isomorphic to $\mathfrak{gl}(m_i, \mathbb{K}_i)$ and $\mathbb{K}_i$ is equal to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ in agreement with the type of the representation $\mathfrak{l}^i$. The action of $(\mathfrak{a}_i \oplus \mathfrak{t}_i)$ on $m_i \cdot \mathfrak{l}^i$ is induced by the canonical action of $\mathfrak{gl}(m_i, \mathbb{K}_i)$ on the span of the maximal weight vectors of each copy of $\mathfrak{l}^i$. Thus we have the cases:

(*i*) $\mathfrak{l}^i$ is of the real or the complex type: then $\mathfrak{a}_i \simeq \mathfrak{sl}(m_i, \mathbb{K}_i)$ and $\mathfrak{t}^i \simeq \mathbb{K}_i$;

(*ii*) $\mathfrak{l}^i$ is of the quaternionic type: then $\mathfrak{a}_i \simeq \mathfrak{sl}(m_i, \mathbb{H}_i) \oplus \mathfrak{su}(2)$ and $\mathfrak{t}^i \simeq \mathbb{R}$.

We summarize the discussion above in the following:

Theorem 3.9. *Let $\mathfrak{s}$ be a transitive semisimple graded CR algebra and $\mathfrak{l}$ a nondegenerate CR graded $\mathfrak{s}$-module.*

Assume that the Levi–Tanaka extension $\mathfrak{g}$ of $\mathfrak{s}$ by $\mathfrak{l}$ is proper.

Let $\mathfrak{l} = \bigoplus m_i \cdot \mathfrak{l}^i$ be a decomposition of $\mathfrak{l}$ into irreducible graded $\mathfrak{s}$-modules. Denote by π_i the projection onto $m_i \cdot \mathfrak{l}^i$. Then $\mathfrak{g}$ is finite dimensional and admits the CR Levi-Malcev decomposition:

$$\mathfrak{g} = (\mathfrak{s} \oplus \bigoplus_i \mathfrak{a}^i) \oplus (\mathfrak{l} \oplus \bigoplus_i \mathfrak{t}^i \oplus \mathfrak{n}),$$

where, for every i, $\mathfrak{a}^i$ and $\mathfrak{t}^i$ are contained in $\mathfrak{g}_0$ and we distinguish the following cases:

(i) *$\mathfrak{l}^i$ is of the real type: then $\mathfrak{a}_i \simeq \mathfrak{sl}(m_i, \mathbb{R})$ and $\mathfrak{t}^i = \mathbb{R}\pi_i$;*

(ii) *$\mathfrak{l}^i$ is of the complex type: then $\mathfrak{a}_i \simeq \mathfrak{sl}(m_i, \mathbb{C})$ and $\mathfrak{t}^i = \mathbb{C}\pi_i$;*

(iii) *$\mathfrak{l}^i$ is of the quaternionic type: then $\mathfrak{a}_i \simeq \mathfrak{sl}(m_i, \mathbb{H}) \oplus \mathfrak{su}(2)$ and $\mathfrak{t}^i = \mathbb{R}\pi_i$.*

Unlike the irreducible case, if $\mathfrak{l}$ is reducible the nilpotent ideal $\mathfrak{n}$ can be different from $\{0\}$:

Example 3.10. Let $\mathfrak{s} = \mathfrak{sl}(3, \mathbb{C})$. It admits an essentially unique structure of a Levi–Tanaka algebra (see [4]), with gradation and partial complex structure defined respectively by the matrices

$$E_{\mathfrak{s}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad J_{\mathfrak{s}} = \begin{pmatrix} \frac{1}{3}i & 0 & 0 \\ 0 & -\frac{2}{3}i & 0 \\ 0 & 0 & \frac{1}{3}i \end{pmatrix}.$$

Then $\mathfrak{s}_-$ is the subalgebra of nilpotent upper triangular matrices, and $\mathfrak{s}_0$ is the Cartan subalgebra of diagonal matrices of $\mathfrak{s}$.

Let $\mathfrak{l}$ be the direct sum of the standard representation $\mathfrak{l}' = \mathbb{C}^3$ with basis $(e_i)_{1 \leq i \leq 3}$ and its dual $\mathfrak{l}'' = (\mathbb{C}^3)^*$ with dual basis $(f_j)_{1 \leq j \leq 3}$. Choose the gradations of $\mathfrak{l}'$ and $\mathfrak{l}''$ by imposing that $\deg e_1 = -3$, $\deg e_2 = -2$, $\deg e_3 = -1$, $\deg f_1 = 0$, $\deg f_2 = -1$ and $\deg f_3 = -2$. Then $\mathfrak{s}_- \oplus \mathfrak{l}_-$ is a fundamental nondegenerate graded Lie algebra. Multiplication by $i = \sqrt{-1}$ defines the CR structure on $\mathfrak{l}$. Denote by $\mathfrak{g} = \mathfrak{g}(\mathfrak{s}_- \oplus \mathfrak{l}_-)$ the corresponding Levi–Tanaka algebra. In this case $\mathfrak{l}$ is an ideal, and via the adjoint representation we can identify $\mathfrak{s} \oplus \mathfrak{t} \oplus \mathfrak{n}$ to a subalgebra of $\mathfrak{gl}(\mathfrak{l}) \simeq \mathfrak{gl}(6, \mathbb{C})$.

Under this identification we have:

$$\mathfrak{s} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix} \middle| A \in \mathfrak{sl}(3, \mathbb{C}) \right\},$$

$$\mathfrak{t} = \left\{ \begin{pmatrix} \alpha I & 0 \\ 0 & \beta I \end{pmatrix} \middle| I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \alpha, \beta \in \mathbb{C} \right\},$$

and

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix} \middle| N \in \mathfrak{o}(3, \mathbb{C}) \right\},$$

graded in such a way that

$$N_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

correspond to elements of degrees 0, 1 and 2, respectively.

3.2. **The CR structure of the Levi–Tanaka extensions.** Our next
goal is to show that the Levi–Tanaka algebras $\mathfrak{g}$ that are prolonga-
tions of nondegenerate graded Abelian CR extensions of semisimple
CR algebras have the J property: this means that the partial complex
structure J on $\mathfrak{g}_{-1}$ is the restriction to $\mathfrak{g}_{-1}$ of the inner derivation with
respect to an element $J_{\mathfrak{g}}$ of $\mathfrak{g}$.

All semisimple graded CR algebras have the J property (see [4]).
Thus we can restrict our consideration to the case where $\mathfrak{g}$ is proper.

If $\mathfrak{s} = \oplus_{p \in \mathbb{Z}} \mathfrak{s}_p$ is a fundamental transitive semisimple graded CR
algebra, we denote by $J_{\mathfrak{s}}$ the element of $\mathfrak{s}_0$ such that $[J_{\mathfrak{s}}, X] = JX$ for
all $X \in \mathfrak{s}_{-1}$.

We shall prove the following:

Theorem 3.11. *Let $\mathfrak{s}$ be a transitive semisimple graded CR algebra.
Let $\mathfrak{l}$ be a nondegenerate transitive CR graded $\mathfrak{s}$-module. Then the
Levi–Tanaka extension $\mathfrak{g}$ of $\mathfrak{s}$ by $\mathfrak{l}$ has the J property.*

The proof will be divided in several steps.

Lemma 3.12. *Let $\mathfrak{s}$ be a transitive semisimple graded CR algebra and
$\mathfrak{l}$ an irreducible nondegenerate graded CR $\mathfrak{s}$-module. Assume that the
Levi–Tanaka extension $\mathfrak{g}$ of $\mathfrak{s}$ by $\mathfrak{l}$ is proper. Then $\mathfrak{g}$ has the J property
if and only if $\mathrm{ad}_{\mathfrak{g}}(J_{\mathfrak{s}})$ acts as a multiple of the identity on $\mathfrak{l}_{-2}$.*

*Moreover if the $\mathfrak{s}$-module $\mathfrak{l}$ is real or quaternionic then $J_{\mathfrak{g}} = J_{\mathfrak{s}}$; if $\mathfrak{l}$
is an $\mathfrak{s}$-module of the complex type then $J_{\mathfrak{g}} = J_{\mathfrak{s}} + ik\pi$, with $k \in \mathbb{R}$.*

Proof. Assume that there is an element $J_{\mathfrak{g}}$ of $\mathfrak{g}$ such that $JX = [J_{\mathfrak{g}}, X]$
for $X \in \mathfrak{g}_{-1} = \mathfrak{s}_{-1} \oplus \mathfrak{l}_{-1}$. Since $\mathrm{ad}_{\mathfrak{g}}(J_{\mathfrak{g}})$ is semisimple (see [3]) and agrees
with $\mathrm{ad}_{\mathfrak{g}}(J_{\mathfrak{s}})$ on $\mathfrak{s}_{-1}$, then $\mathfrak{s}$ is $J_{\mathfrak{g}}$-invariant and we have $[J_{\mathfrak{g}} - J_{\mathfrak{s}}, \mathfrak{s}] = 0$.
Hence, $Z = J_{\mathfrak{g}} - J_{\mathfrak{s}}$ defines an $\mathfrak{s}$-endomorphism of $\mathfrak{l}$ which, by Schur's
lemma, is a multiple of the identity:

$$\mathrm{ad}_{\mathfrak{g}}(Z) = k\pi, \quad k \in \mathbb{K},$$

where $\mathbb{K}$ are either the real or the complex numbers, or the quaternions,
according to the fact that $\mathfrak{l}$ is real, complex or quaternionic.

Assume vice versa that $[J_{\mathfrak{s}}, X] = kX$, for some $k \in \mathbb{K}$, when $X \in \mathfrak{l}_{-2}$,
and define:

$$J_{\mathfrak{g}} = J_{\mathfrak{s}} - k\pi.$$

Then we have, for $X \in \mathfrak{l}_{-1}$ and $Y \in \mathfrak{s}_{-1}$:

$$[[J_{\mathfrak{g}}, X], Y] = [J_{\mathfrak{g}}, [X, Y]] + [X, [J_{\mathfrak{g}}, Y]] = [X, [J_{\mathfrak{s}}, Y]] = [X, JY].$$

This shows that $J_{\mathfrak{g}}$ defines the complex structure of $\mathfrak{g}$, so that $\mathfrak{g}$ has
the J property.

To prove the last statement, we note that, because $[J_{\mathfrak{g}}, \mathfrak{g}_{-2}] = 0$, the
derivation $\mathrm{ad}_{\mathfrak{g}}(J_{\mathfrak{s}})$ acts on $\mathfrak{l}_{-2}$ as $-k \cdot \mathrm{Id}_{\mathfrak{l}_{-2}}$, with $k \in \mathbb{K}$. The restriction
of $\mathrm{ad}_{\mathfrak{g}}(J_{\mathfrak{s}})$ to $\mathfrak{l}$ is an endomorphism with zero trace. Since any two

eigenvalues of $\mathrm{ad}(J_\mathfrak{s})$ differ by an integral multiple of i, all, including k, are purely imaginary.

In particular, when $\mathbb{K} = \mathbb{R}$, we obtain that $J_\mathfrak{g} = J_\mathfrak{s}$.

If $\mathfrak{l}$ is quaternionic, we consider $Z = J_\mathfrak{g} - J_\mathfrak{s}$. The restriction of $\mathrm{ad}_\mathfrak{g}(Z)$ to $\mathfrak{l}$ has purely imaginary eigenvalues, and therefore Z belongs to $\mathfrak{a} \simeq \mathfrak{su}(2)$ (see Theorem 3.8). Since $\mathfrak{a} \subset \mathfrak{g}_0$, we have $[J_\mathfrak{g}, \mathfrak{a}] = 0$ and hence $[Z, \mathfrak{a}] = -[J_\mathfrak{s}, \mathfrak{a}] = 0$ shows that $Z = 0$. $\square$

Assume that both $\mathfrak{s}$ and $\mathfrak{l}$ are complex, i.e. obtained from a complex Lie algebra and a complex representation by restriction of the base field to $\mathbb{R}$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{s}$, with $E_\mathfrak{s} \in \mathfrak{h}$, so that $\mathfrak{h} \subset \mathfrak{s}_0$ and $J_\mathfrak{s} \in \mathfrak{h}$. We shall denote by $\mathcal{R}$ and $\mathcal{P}$ the root system of $\mathfrak{s}$ and the set of weights of $\mathfrak{l}$ with respect to $\mathfrak{h}$, respectively.

The eigenspaces $\mathfrak{s}^\alpha = \{X \in \mathfrak{s} \mid [H, X] = \alpha(H)X \ \forall H \in \mathfrak{h}\} \subset \mathfrak{s}$, for $\alpha \in \mathcal{R}$, and $\mathfrak{l}^\lambda = \{Y \in \mathfrak{l} \mid [H, Y] = \lambda(H)Y \ \forall H \in \mathfrak{h}\} \subset \mathfrak{l}$, for $\lambda \in \mathcal{P}$, consist of homogeneous vectors and therefore we have partitions:

$$\mathcal{R} = \bigcup_{p \in \mathbb{Z}} \mathcal{R}_p \quad \text{and} \quad \mathcal{P} = \bigcup_{p \in \mathbb{Z}} \mathcal{P}_p$$

where $\mathcal{R}_p$ is the set of the roots $\alpha \in \mathcal{R}$ such that $\mathfrak{s}^\alpha \subset \mathfrak{s}_p$ and $\mathcal{P}_p$ that of the weights $\lambda \in \mathcal{P}$ for which $\mathfrak{l}^\lambda \subset \mathfrak{l}_p$. Since $J_\mathfrak{s} \in \mathfrak{h}$, the set $\mathcal{R}_{-1}$ further decomposes into

$$\mathcal{R}_{-1} = \mathcal{R}^{1,0} \cup \mathcal{R}^{0,1}$$

where $\mathcal{R}^{1,0} = \{\alpha \in \mathcal{R}_{-1} \mid \alpha(J_\mathfrak{s}) = i\}$ and $\mathcal{R}^{0,1} = \{\alpha \in \mathcal{R}_{-1} \mid \alpha(J_\mathfrak{s}) = -i\}$. We set

$$\begin{cases} \mathfrak{s}^{1,0} = \{X \in \mathfrak{s}_{-1} \mid [J_\mathfrak{s}, X] = iX\}, \\ \mathfrak{s}^{0,1} = \{X \in \mathfrak{s}_{-1} \mid [J_\mathfrak{s}, X] = -iX\}, \\ \mathfrak{l}^{1,0} = \{Y \in \mathfrak{l}_{-1} \mid JY = iY\}, \\ \mathfrak{l}^{0,1} = \{Y \in \mathfrak{l}_{-1} \mid JY = -iY\}, \\ \mathfrak{g}^{1,0} = \mathfrak{s}^{1,0} \oplus \mathfrak{l}^{1,0}, \\ \mathfrak{g}^{0,1} = \mathfrak{s}^{0,1} \oplus \mathfrak{l}^{0,1}. \end{cases}$$

The integrability condition for the partial complex structure J can be rewritten in the form:

$$(\lozenge) \qquad [\mathfrak{g}^{1,0}, \mathfrak{g}^{1,0}] = \{0\} \quad \text{and} \quad [\mathfrak{g}^{0,1}, \mathfrak{g}^{0,1}] = \{0\}.$$

Then we obtain:

Lemma 3.13. *Let $\lambda \in \mathcal{P}_{-1}$. Then:*

- *If $\alpha \in \mathcal{R}_{-1}$ and $\lambda + \alpha \in \mathcal{P}$, then $\lambda - \alpha \notin \mathcal{P}$;*
- *If $\alpha \in \mathcal{R}_{-1}$ and $\lambda - \alpha \in \mathcal{P}$, then $\lambda + \alpha \notin \mathcal{P}$;*
- *$\dim_\mathbb{C} \mathfrak{l}^\lambda = 1$ and either $\mathfrak{l}^\lambda \subset \mathfrak{l}^{1,0}$, or $\mathfrak{l}^\lambda \subset \mathfrak{l}^{0,1}$.*

Proof. Let $\alpha \in \mathcal{R}_{-1}$ be fixed, and let X_α be a nonzero element of $\mathfrak{s}^\alpha$. If $X_\alpha \in \mathfrak{s}^{1,0}$ and $A \in \mathfrak{l}_0$, then $[X_\alpha, A] \in \mathfrak{l}^{1,0}$. Indeed we have $J([X_\alpha, A]) = [JX_\alpha, A] = [[J_\mathfrak{s}, X_\alpha], A] = i[X_\alpha, A]$. Analogously, if $X_\alpha \in \mathfrak{s}^{0,1}$ and $A \in \mathfrak{l}_0$, then $[X_\alpha, A] \in \mathfrak{l}^{0,1}$. Therefore $[X_\alpha, X_\alpha, A] = 0$ for all $A \in \mathfrak{l}_0$ and $X_\alpha \in \mathfrak{s}^\alpha$ with $\alpha \in \mathcal{R}_{-1}$. It follows that $\mathcal{P}$ does not contain any string of the form $\lambda - \alpha, \lambda, \lambda + \alpha$ with $\alpha \in \mathcal{R}_{-1}$.

On the other hand, since $\mathfrak{s} \oplus \mathfrak{l}$ is nondegenerate, there exists at least one $\alpha \in \mathcal{R}_{-1}$ such that $\lambda + \alpha \in \mathcal{P}$. Thus $\mathcal{P}$ contains a maximal string of the form $\lambda, \lambda + \alpha, \dots, \lambda + p\alpha$, with $p \geq 1$. It follows that $\mathfrak{l}^\lambda$ is one dimensional and, because $\mathfrak{l}^\lambda$ is J-invariant (indeed J commutes with the action of $\mathfrak{h}$ on $\mathfrak{l}_{-1}$) we conclude, from $[X_\alpha, JY] = -[JX_\alpha, Y] = -[[J_\mathfrak{s}, X_\alpha], Y] = -\alpha(J_\mathfrak{s})[X_\alpha, Y]$, that $JY = -\alpha(J_\mathfrak{s})Y$. $\qquad\square$

In particular we obtain a partition

$$\mathcal{P}_{-1} = \mathcal{P}^{1,0} \cup \mathcal{P}^{0,1}$$

where

$$\mathcal{P}^{1,0} = \{\lambda \in \mathcal{P}_{-1} \mid \mathfrak{l}^\lambda \subset \mathfrak{l}^{1,0}\} \quad \text{and} \quad \mathcal{P}^{0,1} = \{\lambda \in \mathcal{P}_{-1} \mid \mathfrak{l}^\lambda \subset \mathfrak{l}^{0,1}\}.$$

Lemma 3.14. *Assume that $\mathfrak{l}$ is irreducible and fix $Y \in \mathfrak{l}_{-2} \setminus \{0\}$. Then $\mathfrak{l}_{-2}$ is generated by the Lie monomials of the form*

$$(*) \qquad\qquad L = [Z_1, \dots, Z_h, X_1, \Xi_1 \dots, X_k, \Xi_k, Y]$$

where $Z_j \in \mathfrak{s}_0$, for $j = 1, ..., h$, and $X_j \in \mathfrak{s}_{-1}$, $\Xi_j \in \mathfrak{s}_1$ for all $j = 1, ..., k$.

Proof. Since $\mathfrak{s}$ is transitive, $\mathfrak{s}_{-1} \cup \mathfrak{s}_1$ is a set of generators of $\mathfrak{s}$. Therefore, the homogeneous Lie monomials $[W_1, \dots, W_\ell, Y]$ with $W_1, \dots, W_\ell \in \mathfrak{s}_{-1} \cup \mathfrak{s}_1$ generate $\mathfrak{l}$. In particular, those of degree 0 generate $\mathfrak{l}_{-2}$. By reordering it is easy to show that these are linear combinations of Lie monomials of the form $(*)$. $\qquad\square$

Lemma 3.15. *Let $\alpha \in \mathcal{R}^{1,0}$ and $\beta \in \mathcal{R}^{0,1}$, and $X_\alpha \in \mathfrak{s}^\alpha$, $\Xi_{-\alpha} \in \mathfrak{s}^{-\alpha}$, $X_\beta \in \mathfrak{s}^\beta$, $\Xi_{-\beta} \in \mathfrak{s}^{-\beta}$, $\Xi \in \mathfrak{s}^{-\beta}$. Then for all $Y \in \mathfrak{l}_{-2}$, we have $[X_\alpha, \Xi_{-\beta}, Y] = [X_\beta, \Xi_{-\alpha}, Y] = 0$.*

Proof. We can assume that $Y \in \mathfrak{l}^\lambda$ for some weight λ. Let us show that $[X_\alpha, \Xi_{-\beta}, Y] = 0$. If $[\Xi_{-\beta}, Y] = 0$, we have nothing to prove. If $[\Xi_{-\beta}, Y] \neq 0$, then $\lambda - \beta$ is a weight of $\mathfrak{l}$. Since the weights of $\mathfrak{l}$ contain the string $\lambda, \lambda - \beta$, there exist $Y' \in \mathfrak{l}^{\lambda - \beta}$ and $X' \in \mathfrak{s}^\beta$ such that

$$[X', Y] \neq 0.$$

As $\mathfrak{s}^\beta \subset \mathfrak{s}^{0,1}$, it follows from equation $(\Diamond)$ that $\mathfrak{l}^{\lambda - \beta} \subset \mathfrak{l}^{1,0}$. Hence $[\Xi_{-\beta}, Y] \in \mathfrak{l}^{1,0}$ and therefore, using again equation $(\Diamond)$, we obtain $[X_\alpha, \Xi_{-\beta}, Y] = 0$.

The other equality is proved in the same way. $\qquad\square$

Proof of Theorem 3.11. The complex case. We can assume that $\mathfrak{g}$ is proper. Suppose first that $\mathfrak{l}$ is irreducible, and let $Y \neq 0$ be an eigenvector of $J_{\mathfrak{s}}$ in $\mathfrak{l}_{-2}$, with $[J_{\mathfrak{s}}, Y] = kY$. By Lemma 3.14, $\mathfrak{l}_{-2}$ is generated by the Lie monomials:

$$L = [Z_1, \ldots, Z_h, X_1, \Xi_1 \ldots, X_k, \Xi_k, Y]$$

where $Z_1, \ldots, Z_h \in \mathfrak{s}_0$ and moreover

$$X_i \in \mathfrak{s}^{\alpha_i} \quad \text{for a root} \quad \alpha_i \in \mathcal{R}_{-1},$$

$$\Xi_i \in \mathfrak{s}^{-\beta_i} \quad \text{for a root} \quad \beta_i \in \mathcal{R}_{-1}.$$

By the preceding lemma, $L = 0$ unless

$$\alpha_i(J_{\mathfrak{s}}) - \beta_i(J_{\mathfrak{s}}) = 0 \quad \text{for every} \quad i = 1, \ldots, k.$$

Then, if $L \neq 0$ we obtain $[J_{\mathfrak{s}}, L] = kL$. This shows that $\mathrm{ad}(J_{\mathfrak{s}})|_{\mathfrak{l}_{-2}}$ is a multiple of the identity and therefore, by Lemma 3.12, $\mathfrak{g}$ has the J property.

If $\mathfrak{l}$ is reducible, $\mathfrak{l} = \bigoplus_{i=1}^{n} \mathfrak{l}_i$, we can apply the argument above to each $\mathfrak{s} \oplus \mathfrak{l}_i$: we find that $J_i = J_{\mathfrak{s}} - k_i \pi_i$ for each i, where π_i is the canonical projection onto $\mathfrak{l}_i$. Then $J_{\mathfrak{g}} = J_{\mathfrak{s}} - \sum_{i=1}^{n} k_i \pi_i \in \mathfrak{g}$ defines the partial complex structure of $\mathfrak{s} \oplus \mathfrak{l}$.

The general case. If $\mathfrak{l}$ contains irreducible $\mathfrak{s}$-submodules that are not of the complex type, we apply the argument above to the complexification $\hat{\mathfrak{g}} = \mathfrak{g}(\hat{\mathfrak{s}} \oplus \hat{\mathfrak{l}})$. In particular we obtain that $\mathrm{ad}(J_{\mathfrak{s}})$ is a multiple of the identity on the subspace of degree -2 of each graded irreducible component of $\hat{\mathfrak{l}}$. If $\mathfrak{l}^{(i)}$ is a graded irreducible component of $\mathfrak{l}$ that is of the real type, then its complexification $\hat{\mathfrak{l}}^{(i)}$ is an irreducible component of $\hat{\mathfrak{l}}$. Then by the invariance under conjugation it follows that $\mathrm{ad}(J_{\mathfrak{s}})$ is actually zero on $\mathfrak{l}_{-2}^{(i)}$.

In the same way, if $\mathfrak{l}^{(i)}$ is a graded irreducible component of $\mathfrak{l}$ that is of the quaternionic type, the complexification $\hat{\mathfrak{l}}^{(i)}$ is the direct sum of two isomorphic simple complex $\hat{\mathfrak{s}}$-modules $\mathfrak{v}$ and $\bar{\mathfrak{v}}$, that are exchanged by the conjugation. Since $\mathrm{ad}(J_{\mathfrak{s}})$ defines on $\mathfrak{v}_{-2}$ and $\bar{\mathfrak{v}}_{-2}$ the same purely imaginary multiple of the identity, this must coincide with its conjugate. Thus it is zero and the partial complex structure J on $\mathfrak{l}_{-1}^{(i)}$ is defined by the restriction to $\mathfrak{l}_{-1}^{(i)}$ of $\mathrm{ad}(J_{\mathfrak{s}})$.

This shows that, for the element $J_{\mathfrak{g}}$, obtained for the complexification $\hat{\mathfrak{g}} = \mathfrak{g}(\hat{\mathfrak{s}} \oplus \hat{\mathfrak{l}})$ using the argument in the first part of the proof, we have $\mathrm{ad}(J_{\mathfrak{g}}) = \mathrm{ad}(J_{\mathfrak{s}})$ on the complexifications $\hat{\mathfrak{l}}^{(i)}$ of all graded irreducible components $\mathfrak{l}^{(i)}$ that are of the real or of the quaternionic type. Therefore $J_{\mathfrak{g}}$ belongs to $\mathfrak{g}(\mathfrak{s} \oplus \mathfrak{l})$. This completes the proof. $\square$

Example 3.16. Let $\mathfrak{s} = \mathfrak{sl}(2, \mathbb{C})$, with the same gradation and CR structure as in Example 3.4. Let $n \geq 2$ and let $\mathfrak{l}^n$ be the irreducible representation of $\mathfrak{s}$ of dimension n. Lemma 3.13 implies that there exists a unique gradation and CR structure on $\mathfrak{l}^n$ that makes it into

a nondegenerate graded CR representation of $\mathfrak{s}$. It is obtained by assigning degree -1 to the minimal weight vector, so that $\mathfrak{l}^n = \mathfrak{l}^n_-$.

The algebra $\mathfrak{s} \oplus \mathfrak{l}^n$ is a partial Levi–Tanaka extension if $n \geq 2$. It is semisimple, with $\mathfrak{g}(\mathfrak{s} \oplus \mathfrak{l}^2) \simeq \mathfrak{sl}(3, \mathbb{C})$, if $n = 2$, and it is proper, with $\mathfrak{g}(\mathfrak{s} \oplus \mathfrak{l}^n) = \mathfrak{s} \oplus \mathfrak{l}^n \oplus \mathbb{C}\pi$, if $n \geq 3$.

3.3. CR **admissibility of a graded representation.** Let $\mathfrak{s}$ be a semisimple graded CR algebra. In this section we investigate the conditions for a graded CR module $\mathfrak{l}$ to be a CR representation of $\mathfrak{s}$.

Recall that, if $\mathfrak{s}$ is Levi–Tanaka, all Levi–Tanaka extensions of $\mathfrak{s}$ are proper.

Lemma 3.17. *Let $\mathfrak{s}$ be a fundamental graded Lie algebra and $\mathfrak{l} = \oplus_{p \geq -\nu}\mathfrak{l}_p$, with $\mathfrak{l}_{-\nu} \neq \{0\}$, a graded irreducible representation of $\mathfrak{s}$. Then for every $Y \in \mathfrak{l} \setminus \mathfrak{l}_{-\nu}$ there exists $X \in \mathfrak{s}_{-1}$ such that $[X, Y] \neq 0$.*

Proof. It suffices to consider a $Y \in \mathfrak{l}_p \setminus \{0\}$, with $p > -\nu$. From Lemma 2.1 we know that $\mathfrak{l}$ is generated by decreasing homogeneous Lie monomials in Y. Since $\mathfrak{l}_{-\nu} \neq \{0\}$, there are homogeneous elements $Z_1, \ldots, Z_k$ of $\mathfrak{s}_-$ such that $0 \neq [Z_1, \ldots, Z_k, Y] \in \mathfrak{l}_{-\nu}$. In particular $[Z_k, Y] \neq 0$ and, because $\mathfrak{s}_-$ is generated by $\mathfrak{s}_{-1}$, there is X in $\mathfrak{s}_{-1}$ such that $[X, Y] \neq 0$. □

We have the following:

Proposition 3.18. *A partial Levi–Tanaka extension of a semisimple graded CR algebra $\mathfrak{s}$ is completely determined by the data of a representation $\mathfrak{l}$, of a decomposition $\mathfrak{l} = \oplus\mathfrak{l}^{(i)}$ of $\mathfrak{l}$ into a direct sum of irreducible $\mathfrak{s}$ modules, and, for each irreducible component $\mathfrak{l}^{(i)}$, of its kind $\mu_i = \mu(\mathfrak{l}^{(i)})$.*

Vice versa, given a semisimple Levi–Tanaka algebra $\mathfrak{s}$ and a graded representation $\mathfrak{l}$, the extension of $\mathfrak{s}$ by $\mathfrak{l}$ is a partial Levi–Tanaka extension if and only if every irreducible graded component $\mathfrak{l}^{(i)}$ of $\mathfrak{l}$ satisfies the following conditions:

(i) $\mathfrak{l}^{(i)}_{-1} \neq 0$, *i.e.* $\nu(\mathfrak{l}^{(i)}) \geq -1$;
(ii) $\mathfrak{l}^{(i)}_{-2} \neq 0$, *i.e.* $\mu(\mathfrak{l}^{(i)}) \geq 2$;
(iii) $\mathfrak{l}^{(i)}$ *is CR.*

Proof. The first statement is a direct consequence of Lemma 2.4 and Lemma 2.6.

Conditions (i), (ii) and (iii) in the second statement are clearly necessary, and the previous lemma tells us that, when conditions (i) and (ii) are satisfied, $\mathfrak{s} \oplus \mathfrak{l}$ is transitive and nondegenerate.

We only need to check that in this case $\mathfrak{l}$ is also fundamental, and to this aim it suffices to consider the case where $\mathfrak{l}$ is irreducible.

Suppose that $\mathfrak{l}$ is irreducible and fix $Y \in \mathfrak{l}_{-1}$. By Lemma 2.1, the subspace $\oplus_{p < -1}\mathfrak{l}_p$ is generated by homogeneous Lie monomials in Y. Since $\mathfrak{s}$ is generated by its elements of degree 0, 1 and -1, we can use

as generators homogeneous Lie monomials that only contain elements of these degrees.

Rearranging their terms, we obtain a set of generators of $\oplus_{p<-1}\mathfrak{l}_p$ that are homogeneous Lie monomials of the form:

$$[X_1, \ldots, X_k, \Xi_1, \ldots, \Xi_l, A_1, \ldots, A_h, Y]$$

where $X_i \in \mathfrak{s}_{-1}$, $\Xi_i \in \mathfrak{s}_1$ and $A_i \in \mathfrak{s}_0$ for all i, and $k > l + 1$. Every such monomial can be written as

$$[X_1, \ldots, X_{k-l}, W]$$

where

$$W = [X_{k-l+1}, \ldots, X_k, \Xi_1, \ldots, \Xi_l, A_1, \ldots, A_h, Y] \in \mathfrak{l}_{-1}. \qquad \square$$

By the previous proposition, to characterize the homogeneous CR representations $\mathfrak{l}$ of $\mathfrak{s}$, it will suffice to restrict our consideration to the case of an irreducible $\mathfrak{l}$.

Proposition 3.19. *Let $\mathfrak{s}$ be a semisimple graded CR algebra and $\mathfrak{l}$ a graded irreducible representation of $\mathfrak{s}$. Then $\mathfrak{s} \oplus \mathfrak{l}$ is a partial Levi–Tanaka extension of $\mathfrak{s}$ if and only if*

- *(i) $\mathfrak{l}_{-1} \neq 0$;*
- *(ii) $\mathfrak{l}_{-2} \neq 0$;*
- *(iii) there exists $k \in \mathbb{R}$ such that $\mathrm{ad}(J_\mathfrak{s} - ik\pi) = 0$ on $\mathfrak{l}_0 \oplus \mathfrak{l}_{-2}$.*

Proof. The conditions are necessary in view of Proposition 3.18, Theorem 3.11 and Lemma 3.12, since by Lemma 3.11 in [3], the element $J_\mathfrak{g}$ of the Levi–Tanaka prolongation of $\mathfrak{s} \oplus \mathfrak{l}$ that defines the partial complex structure satisfies $[J_\mathfrak{g}, \mathfrak{l}_0 \oplus \mathfrak{l}_{-2}] = \{0\}$.

Assume vice versa that conditions $(i),(ii)$ and (iii) are satisfied and define (we keep for π and $\mathfrak{a}$ the notation of the preceding sections) $\tilde{J} = J_\mathfrak{s} - ik\pi \in \mathfrak{s} \oplus \mathfrak{l} \oplus \mathfrak{a}$. Then

$$[\tilde{J}, X, Y] = [[\tilde{J}, X], Y] + [X, [\tilde{J}, Y]] = \begin{cases} 0 & \text{if } X \in \mathfrak{s}_{-1}, \ Y \in \mathfrak{l}_{-1}; \\ [X, [\tilde{J}, Y]] & \text{if } X \in \mathfrak{s}_0, \ Y \in \mathfrak{l}_{-1}; \\ [JX, Y] & \text{if } X \in \mathfrak{s}_{-1}, \ Y \in \mathfrak{l}_0; \end{cases}$$

and, for $X \in \mathfrak{s}_{-1}$, $Y \in \mathfrak{l}_{-1}$,

$$[X, [\tilde{J}, [\tilde{J}, Y]]] = [\tilde{J}, [X, [\tilde{J}, Y]]] - [JX, [\tilde{J}, Y]] =$$
$$= 0 - [\tilde{J}, [JX, Y]] + [JJX, Y] = -[X, Y].$$

Thus $\mathrm{ad}(\tilde{J})$ defines a partial complex structure on $\mathfrak{s} \oplus \mathfrak{l}$ and, by Proposition 3.18, $\mathfrak{s} \oplus \mathfrak{l}$ is a partial Levi–Tanaka extension of $\mathfrak{s}$. $\qquad \square$

Finally we give a characterization of the graded CR representations $\mathfrak{l}$ of a semisimple graded CR algebra $\mathfrak{s}$ in terms of the weights of the (complexification of the) $\mathfrak{s}$-module $\mathfrak{l}$.

We recall from [3] that the complexification of a Levi–Tanaka algebra is still a Levi–Tanaka algebra and that vice versa a Levi–Tanaka algebra structure on the complexification $\hat{\mathfrak{g}}$ of a graded Lie algebra $\mathfrak{g} = \oplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ induces a Levi–Tanaka structure on $\mathfrak{g}$ if and only if $\mathfrak{g}_{-1}$ is invariant for the complex structure J of $\hat{\mathfrak{g}}_{-1}$.

We use the same notation of Section 3.2. Moreover, we denote by $\mathcal{B}$ a system of simple vectors of $\mathcal{R}$, such that the eigenspaces of the negative roots are contained in $\hat{\mathfrak{s}}_+$ and we have a partition $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}^{1,0} \cup \mathcal{B}^{0,1}$, with $\mathcal{B}_0 = \mathcal{B} \cap \mathcal{R}_0$, $\mathcal{B}^{1,0} = \mathcal{B} \cap \mathcal{R}^{1,0}$, $\mathcal{B}^{0,1} = \mathcal{B} \cap \mathcal{R}^{0,1}$.

Theorem 3.20. *Let $\mathfrak{s}$ be a semisimple graded CR algebra of the complex type and $\mathfrak{l}$ a complex irreducible graded representation of $\mathfrak{s}$, such that $\mathfrak{l}_{-1} \neq 0$ and $\mathfrak{l}_{-2} \neq 0$. Then the following conditions are equivalent:*

(i) *$\mathfrak{s} \oplus \mathfrak{l}$ is a partial Levi–Tanaka extension of $\mathfrak{s}$;*

(ii) *$\mathrm{ad}(J_\mathfrak{s})$ acts as a scalar multiple of the identity on $\mathfrak{l}_0 \oplus \mathfrak{l}_{-2}$;*

(iii) *if $\lambda \in \mathcal{P}_{-2}$, $\alpha, \alpha' \in \mathcal{R}^{1,0}$, $\beta, \beta' \in \mathcal{R}^{0,1}$, then none of the weights $\lambda - \alpha - \alpha'$, $\lambda - \alpha + \beta$, $\lambda - \beta - \beta'$, $\lambda + \alpha - \beta$ belongs to $\mathcal{P}$;*

(iv) *There exists a partition $\mathcal{P}_{-1} = \mathcal{P}^{1,0} \cup \mathcal{P}^{0,1}$ such that*
   (a) *$(\mathcal{P}^{1,0} \pm \mathcal{B}_0) \cap \mathcal{P} \subset \mathcal{P}^{1,0}$ and $(\mathcal{P}^{0,1} \pm \mathcal{B}_0) \cap \mathcal{P} \subset \mathcal{P}^{0,1}$,*
   (b) *$(\mathcal{P}^{1,0} + \mathcal{B}^{1,0}) \cap \mathcal{P} = \emptyset$, $(\mathcal{P}^{1,0} - \mathcal{B}^{0,1}) \cap \mathcal{P} = \emptyset$, $(\mathcal{P}^{0,1} + \mathcal{B}^{0,1}) \cap \mathcal{P} = \emptyset$, $(\mathcal{P}^{0,1} - \mathcal{B}^{1,0}) \cap \mathcal{P} = \emptyset$.*

Proof. $(i) \Leftrightarrow (ii)$.   This is Proposition 3.19 in the complex case.

$(ii) \Rightarrow (iii)$.   In fact we obtain $(\lambda - \alpha - \alpha')(J_\mathfrak{s}) = \lambda(J_\mathfrak{s}) - 2i$, $(\lambda - \beta - \beta')(J_\mathfrak{s}) = \lambda(J_\mathfrak{s}) + 2i$, $(\lambda - \alpha + \beta)(J_\mathfrak{s}) = \lambda(J_\mathfrak{s}) - 2i$, $(\lambda + \alpha - \beta)(J_\mathfrak{s}) = \lambda(J_\mathfrak{s}) + 2i$. If any of these weights belongs to $\mathcal{P}$, statement (ii) is contradicted.

$(iii) \Rightarrow (iv)$.   For each $\lambda \in \mathcal{P}_{-2}$ define $\mathcal{P}^{0,1}_\lambda = (\lambda - \mathcal{R}^{1,0}) \cap \mathcal{P}$ and $\mathcal{P}^{1,0}_\lambda = (\lambda - \mathcal{R}^{0,1}) \cap \mathcal{P}$. Let $\mathcal{P}^{0,1} = \bigcup_\lambda \mathcal{P}^{0,1}_\lambda$ and $\mathcal{P}^{1,0} = \bigcup_\lambda \mathcal{P}^{1,0}_\lambda$. This is a partition of $\mathcal{P}_{-1}$. Indeed every weight in $\mathcal{P}_{-1}$ is contained either in $\mathcal{P}^{0,1}$ or in $\mathcal{P}^{1,0}$, because $\mathfrak{l}$ is nondegenerate. The fact that $\mathcal{P}^{0,1} \cap \mathcal{P}^{1,0} = \emptyset$ and conditions (iv,a) and (iv,b) follow from condition (iii).

$(iv) \Rightarrow (i)$.   Define a complex structure J on $\mathfrak{l}_{-1}$ by imposing that J has eigenvalues i and $-i$ on weight spaces corresponding to weights in $\mathcal{P}^{1,0}$ and $\mathcal{P}^{0,1}$, respectively. This defines a complex structure that is compatible with the CR structure of $\mathfrak{s}$ and thus defines a CR structure on $\mathfrak{l}$. Proposition 3.18 then implies that $\mathfrak{s} \oplus \mathfrak{l}$ is a Levi–Tanaka extension of $\mathfrak{s}$. $\square$

4. Examples

In this section we give some examples of the Levi–Tanaka extensions of semisimple graded CR algebras.

4.1. **The Standard manifold of a Levi–Tanaka extension.** To every Levi–Tanaka algebra $\mathfrak{g}$ corresponds a homogeneous CR manifold $M(\mathfrak{g})$, called the *standard CR manifold* associated to $\mathfrak{g}$ (see [5] for more

details). To construct $M(\mathfrak{g})$, consider the simply connected Lie group G with Lie algebra $\mathfrak{g}$ and let G_+ be the connected analytic subgroup of G whose Lie algebra is $\mathfrak{g}_+$. The analytic subgroup G_+ is closed, and $M(\mathfrak{g}) = G/G_+$ is a smooth manifold. The tangent space at $o = eG_+$ is isomorphic to $\mathfrak{g}/\mathfrak{g}_+$ and can be identified with $\mathfrak{g}_-$. In this way a partial complex structure is defined on $T_o M$; via left translations by the elements of G one defines on $M(\mathfrak{g})$ a G-invariant CR structure. In this way we obtain a complete, Levi-nondegenerate CR manifold, for which G is exactly the identity component of the group of CR automorphisms.

Furthermore $M(\mathfrak{g})$ is compact if and only if $\mathfrak{g}$ is semisimple.

In this context Levi–Tanaka extensions correspond to homogeneous vector bundles: denote by S and S_+ the connected analytic subgroups of G corresponding to $\mathfrak{s}$ and $\mathfrak{s}_+$. In [5] it is proved that G/G_+ is a Mostow fibration on S/S_+. The fiber over $o = eS_+$ is naturally identified with $\mathfrak{l}/\mathfrak{l}_+$. Being $\mathfrak{l}$ a linear representation of $\mathfrak{s}$, and consequently of S, G/G_+ has a canonical structure of an S-homogeneous vector bundle over S/S_+ and the isomorphism

$$G/G_+ \simeq S \times_{S_+} (\mathfrak{l}/\mathfrak{l}_+)$$

is S-equivariant. The isomorphism $\mathfrak{l}/\mathfrak{l}_+ \simeq \mathfrak{l}_-$ induces a partial complex structure on every fiber that coincides with the CR structure induced by the inclusion in G/G_+.

4.2. The adjoint representation I.

If $\mathfrak{l}$ is the adjoint representation of $\mathfrak{s}$ then $\mathfrak{s} \oplus \mathfrak{l}$ is always a partial Levi–Tanaka extension of $\mathfrak{s}$ (the proof is straightforward) and the resulting vector bundle is isomorphic to the tangent bundle of $S/S_+ = M(\mathfrak{s})$, with the natural CR structure.

4.3. The adjoint representation II.

If all simple ideals of $\mathfrak{s}$ are fundamental graded transitive CR algebras (not necessarily Levi–Tanaka), then the adjoint action of $J_\mathfrak{s}$ defines a complex structure on $\mathfrak{s}_1$, because the eigenvalues of $J_\mathfrak{s}$ on $\mathfrak{s}_1$ can only be $\pm i$.

If $X \in \mathfrak{s}_2$ and $Y \in \mathfrak{s}_{-2}$ then

$$\kappa([J_\mathfrak{s}, X], Y) = -\kappa(X, [J_\mathfrak{s}, Y]) = 0$$

where κ denotes the Killing form, and hence $[J_\mathfrak{s}, \mathfrak{s}_2] = 0$. Furthermore, $[J_\mathfrak{s}, \mathfrak{s}_0] = 0$ because the action of $\mathfrak{s}_0$ is compatible with the CR structure.

Let $\mathfrak{l}$ be the adjoint representation, with a "shifted" gradation: $\mathfrak{l}_d \simeq \mathfrak{s}_{d+2}$. The relations above imply that $\mathrm{ad}(J_\mathfrak{s})$ defines a CR structure on $\mathfrak{l}$, that is compatible with the action of $\mathfrak{s}$. It is easy to check that the resulting algebra is transitive, fundamental and nondegenerate. Therefore $\mathfrak{s} \oplus \mathfrak{l}$ is a partial Levi–Tanaka extension of $\mathfrak{s}$.

4.4. **Levi–Tanaka extensions of $\mathfrak{su}(1,2)$.** In this subsection we classify all Levi–Tanaka extensions of $\mathfrak{su}(1,2)$. To this aim, passing to complexifications, we first classify all complex Levi–Tanaka extensions of $\hat{\mathfrak{s}} = \mathfrak{sl}(3,\mathbb{C})$.

We know from [4] that $\hat{\mathfrak{s}}$ admits an essentially unique structure of Levi–Tanaka algebra. More precisely, the gradation and the complex structure are defined by the inner derivations associated to the matrices:

$$
E_{\hat{\mathfrak{s}}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \text{and} \qquad J_{\hat{\mathfrak{s}}} = \begin{pmatrix} \frac{1}{3}i & 0 & 0 \\ 0 & -\frac{2}{3}i & 0 \\ 0 & 0 & \frac{1}{3}i \end{pmatrix}.
$$

Fix the Cartan subalgebra $\hat{\mathfrak{h}}$ of diagonal matrices. Let $\mathcal{B} = \{\alpha_1, \alpha_2\}$ be the standard basis of the root system $\mathcal{R}$ of $\hat{\mathfrak{s}}$. Roots and weights of $\hat{\mathfrak{s}}$ are all elements of a two dimensional Euclidean space, that we identify with the standard Euclidean space $\mathbb{R}^2$. If ω_1 and ω_2 are the fundamental dominant weights, every irreducible representation $\hat{\mathfrak{l}}$ is characterized by its maximal weight ω, that can be written as $\omega = k_1\omega_1 + k_2\omega_2$, with k_1, k_2 non negative integers; the corresponding representation $\hat{\mathfrak{l}}$ is denoted by Γ_{k_1,k_2}. Figure 1 shows the basis $\mathcal{B}$ and the fundamental dominant weights.
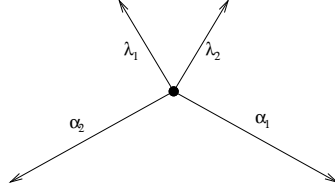

FIGURE 1. Roots and weights of $\mathfrak{sl}(3,\mathbb{C})$

The representation $\Gamma_{1,0}$ is the standard representation of $\mathfrak{sl}(3,\mathbb{C})$. Its weight diagram is depicted in Figure 2
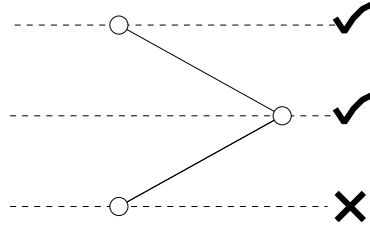

FIGURE 2. Weights of $\Gamma_{1,0}$

In our pictures weights of the same degree will lay on a same dotted line; we put a mark on the right of a dotted line if there is a partial Levi–Tanaka extension by $\hat{\mathfrak{l}}$ in which that line corresponds to degree -1.

For $\mathfrak{sl}(3,\mathbb{C})$ the equivalent conditions of Theorem 3.20 can be graphically described by saying that none of the configurations of Figure 3 could appear in the weight diagram of $\hat{\mathfrak{l}}$.
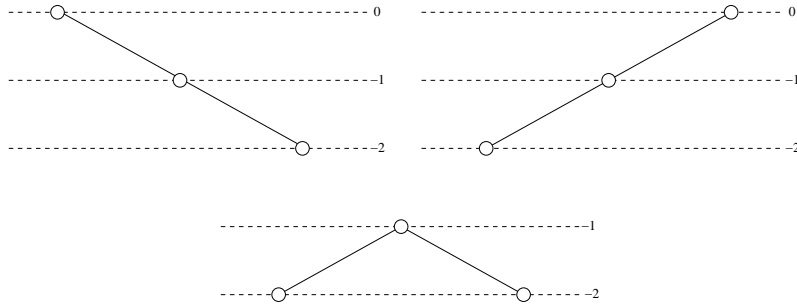

FIGURE 3. "Forbidden" configurations

The situation for $\Gamma_{2,0}$ is depicted in Figure 4. For representations of type $\Gamma_{n,0}$ with $n \geq 3$ it is possible to satisfy the conditions of Theorem 3.20 only by assigning degree -1 to the homogeneous subspace of maximal degree (see Figure 5).
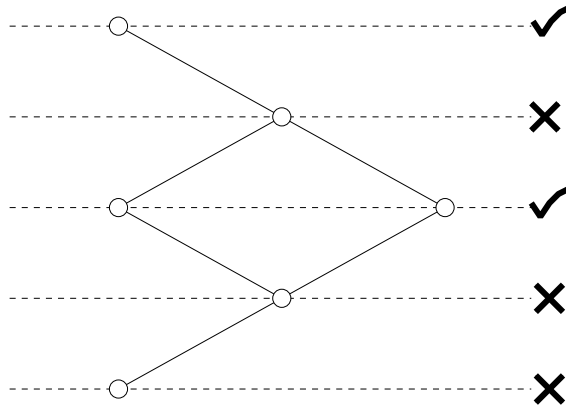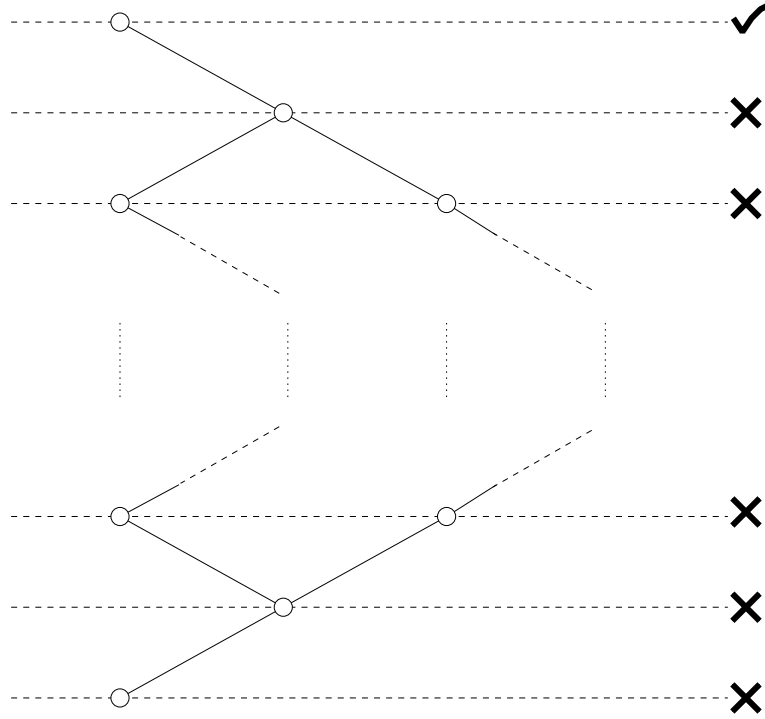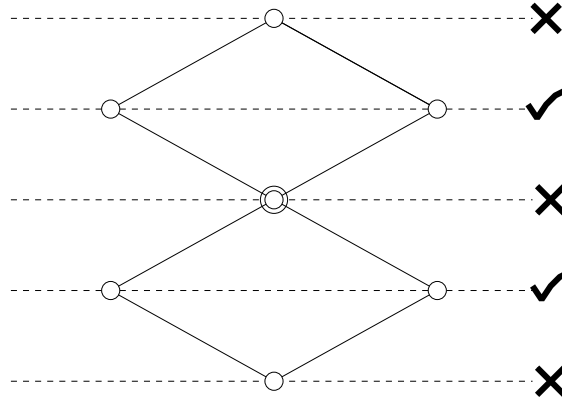

FIGURE 4. Weights of $\Gamma_{2,0}$

Representations of type $\Gamma_{0,n}$ are analogous to those of type $\Gamma_{n,0}$.

The only representation Γ_{k_1,k_2} with k_1 and k_2 both positive that admits a CR structure is the adjoint representation $\Gamma_{1,1}$. In this case there are exactly two distinct CR structures , that are those described in the previous subsections and are depicted in Figure 6.

Now we apply these results to the case of the real algebra $\mathfrak{s} = \mathfrak{su}(1,2)$. To this aim, we need to consider the CR admissible complex representations of $\hat{\mathfrak{s}}$ as real representations of $\mathfrak{s}$. While $\Gamma_{1,1}$ splits into two copies of the adjoint representation $\mathfrak{l}_{ad}$, the representations $\Gamma_{n,0}$ and $\Gamma_{0,n}$ remain irreducible. Hence the irreducible partial Levi–Tanaka extensions of $\mathfrak{s}$ are $\Gamma_{n,0}$, $\Gamma_{0,n}$ and the adjoint representation, with the

FIGURE 5. Weights of $\Gamma_{n,0}$

FIGURE 6. Weights of $\Gamma_{1,1}$

gradations described above and the corresponding unique compatible CR structures.

The representations $\Gamma_{n,0}$ and $\Gamma_{0,n}$ are of the complex type, the adjoint representation $\mathfrak{l}_{\mathrm{ad}}$ is of the real type. Their corresponding Levi–Tanaka extensions are:

$$\mathfrak{s} \oplus \Gamma_{n,0} \oplus \mathbb{C}\pi,$$
$$\mathfrak{s} \oplus \Gamma_{0,n} \oplus \mathbb{C}\pi,$$
$$\mathfrak{s} \oplus \mathfrak{l}_{\mathrm{ad}} \oplus \mathbb{R}\pi.$$

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